

Linear Momentum and Collisions



CHAPTER OUTLINE

- 9.1 Linear Momentum and Its Conservation
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- 9.3 Collisions in One Dimension
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- 9.5 The Center of Mass
- 9.6 Motion of a System of Particles
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▲ A moving bowling ball carries momentum, the topic of this chapter. In the collision between the ball and the pins, momentum is transferred to the pins. (Mark Cooper/Corbis Stock Market)



Consider what happens when a bowling ball strikes a pin, as in the opening photograph. The pin is given a large velocity as a result of the collision; consequently, it flies away and hits other pins or is projected toward the backstop. Because the average force exerted on the pin during the collision is large (resulting in a large acceleration), the pin achieves the large velocity very rapidly and experiences the force for a very short time interval. According to Newton's third law, the pin exerts a reaction force on the ball that is equal in magnitude and opposite in direction to the force exerted by the ball on the pin. This reaction force causes the ball to accelerate, but because the ball is so much more massive than the pin, the ball's acceleration is much less than the pin's acceleration.

Although F and a are large for the pin, they vary in time—a complicated situation! One of the main objectives of this chapter is to enable you to understand and analyze such events in a simple way. First, we introduce the concept of *momentum*, which is useful for describing objects in motion. Imagine that you have intercepted a football and see two players from the opposing team approaching you as you run with the ball. One of the players is the 180-lb quarterback who threw the ball; the other is a 300-lb lineman. Both of the players are running toward you at 5 m/s. However, because the two players have different masses, intuitively you know that you would rather collide with the quarterback than with the lineman. The momentum of an object is related to both its mass and its velocity. The concept of momentum leads us to a second conservation law, that of conservation of momentum. This law is especially useful for treating problems that involve collisions between objects and for analyzing rocket propulsion. In this chapter we also introduce the concept of the center of mass of a system of particles. We find that the motion of a system of particles can be described by the motion of one representative particle located at the center of mass.

9.1 Linear Momentum and Its Conservation

In the preceding two chapters we studied situations that are complex to analyze with Newton's laws. We were able to solve problems involving these situations by applying a conservation principle—conservation of energy. Consider another situation—a 60-kg archer stands on frictionless ice and fires a 0.50-kg arrow horizontally at 50 m/s. From Newton's third law, we know that the force that the bow exerts on the arrow will be matched by a force in the opposite direction on the bow (and the archer). This will cause the archer to begin to slide backward on the ice. But with what speed? We cannot answer this question directly using *either* Newton's second law or an energy approach—there is not enough information.

Despite our inability to solve the archer problem using our techniques learned so far, this is a very simple problem to solve if we introduce a new quantity that describes motion, *linear momentum*. Let us apply the General Problem-Solving Strategy and *conceptualize* an isolated system of two particles (Fig. 9.1) with masses m_1 and m_2 and moving with velocities \mathbf{v}_1 and \mathbf{v}_2 at an instant of time. Because the system is isolated, the only force on

one particle is that from the other particle and we can *categorize* this as a situation in which Newton's laws will be useful. If a force from particle 1 (for example, a gravitational force) acts on particle 2, then there must be a second force—equal in magnitude but opposite in direction—that particle 2 exerts on particle 1. That is, they form a Newton's third law action–reaction pair, so that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can express this condition as

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

Let us further *analyze* this situation by incorporating Newton's second law. Over some time interval, the interacting particles in the system will accelerate. Thus, replacing each force with $m\mathbf{a}$ gives

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$$

Now we replace the acceleration with its definition from Equation 4.5:

$$m_1 \frac{d\mathbf{v}_1}{dt} + m_2 \frac{d\mathbf{v}_2}{dt} = 0$$

If the masses m_1 and m_2 are constant, we can bring them into the derivatives, which gives

$$\begin{aligned} \frac{d(m_1\mathbf{v}_1)}{dt} + \frac{d(m_2\mathbf{v}_2)}{dt} &= 0 \\ \frac{d}{dt}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) &= 0 \end{aligned} \quad (9.1)$$

To *finalize* this discussion, note that the derivative of the sum $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ with respect to time is zero. Consequently, this sum must be constant. We learn from this discussion that the quantity $m\mathbf{v}$ for a particle is important, in that the sum of these quantities for an isolated system is conserved. We call this quantity *linear momentum*:

The **linear momentum** of a particle or an object that can be modeled as a particle of mass m moving with a velocity \mathbf{v} is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v} \quad (9.2)$$

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity \mathbf{v} . Its direction is along \mathbf{v} , it has dimensions ML/T, and its SI unit is kg·m/s.

If a particle is moving in an arbitrary direction, \mathbf{p} must have three components, and Equation 9.2 is equivalent to the component equations

$$p_x = mv_x \quad p_y = mv_y \quad p_z = mv_z$$

As you can see from its definition, the concept of momentum¹ provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product $m\mathbf{v}$ *quantity of motion*; this is perhaps a more graphic description than our present-day word *momentum*, which comes from the Latin word for movement.

Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle. We start with Newton's second law and substitute the definition of acceleration:

$$\Sigma \mathbf{F} = m\mathbf{a} = m \frac{d\mathbf{v}}{dt}$$

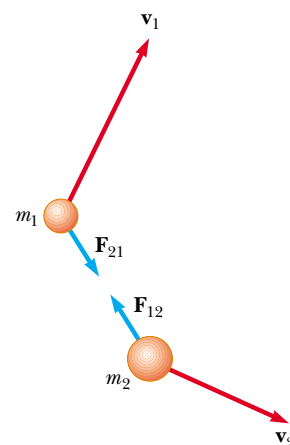


Figure 9.1 Two particles interact with each other. According to Newton's third law, we must have $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Definition of linear momentum of a particle

¹ In this chapter, the terms *momentum* and *linear momentum* have the same meaning. Later, in Chapter 11, we shall use the term *angular momentum* when dealing with rotational motion.

Newton's second law for a particle

In Newton's second law, the mass m is assumed to be constant. Thus, we can bring m inside the derivative notation to give us

$$\Sigma \mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt} \quad (9.3)$$

This shows that **the time rate of change of the linear momentum of a particle is equal to the net force acting on the particle.**

This alternative form of Newton's second law is the form in which Newton presented the law and is actually more general than the form we introduced in Chapter 5. In addition to situations in which the velocity vector varies with time, we can use Equation 9.3 to study phenomena in which the mass changes. For example, the mass of a rocket changes as fuel is burned and ejected from the rocket. We cannot use $\Sigma \mathbf{F} = m\mathbf{a}$ to analyze rocket propulsion; we must use Equation 9.3, as we will show in Section 9.7.

The real value of Equation 9.3 as a tool for analysis, however, arises if we apply it to a *system* of two or more particles. As we have seen, this leads to a law of conservation of momentum for an isolated system. Just as the law of conservation of energy is useful in solving complex motion problems, the law of conservation of momentum can greatly simplify the analysis of other types of complicated motion.

Quick Quiz 9.1 Two objects have equal kinetic energies. How do the magnitudes of their momenta compare? (a) $p_1 < p_2$ (b) $p_1 = p_2$ (c) $p_1 > p_2$ (d) not enough information to tell.

Quick Quiz 9.2 Your physical education teacher throws a baseball to you at a certain speed, and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. You are given the following choices: You can have the medicine ball thrown with (a) the same speed as the baseball (b) the same momentum (c) the same kinetic energy. Rank these choices from easiest to hardest to catch.

Using the definition of momentum, Equation 9.1 can be written

$$\frac{d}{dt} (\mathbf{p}_1 + \mathbf{p}_2) = 0$$

Because the time derivative of the total momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2$ is *zero*, we conclude that the *total* momentum of the system must remain constant:

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \quad (9.4)$$

or, equivalently,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \quad (9.5)$$

where \mathbf{p}_{1i} and \mathbf{p}_{2i} are the initial values and \mathbf{p}_{1f} and \mathbf{p}_{2f} the final values of the momenta for the two particles for the time interval during which the particles interact. Equation 9.5 in component form demonstrates that the total momenta in the x , y , and z directions are all independently conserved:

$$p_{ix} = p_{fx} \quad p_{iy} = p_{fy} \quad p_{iz} = p_{fz} \quad (9.6)$$

This result, known as the **law of conservation of linear momentum**, can be extended to any number of particles in an isolated system. It is considered one of the most important laws of mechanics. We can state it as follows:

PITFALL PREVENTION

9.1 Momentum of a System is Conserved

Remember that the momentum of an isolated *system* is conserved. The momentum of one particle within an isolated system is not necessarily conserved, because other particles in the system may be interacting with it. Always apply conservation of momentum to an isolated *system*.

Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

Conservation of momentum

This law tells us that **the total momentum of an isolated system at all times equals its initial momentum.**

Notice that we have made no statement concerning the nature of the forces acting on the particles of the system. The only requirement is that the forces must be *internal* to the system.

Quick Quiz 9.3 A ball is released and falls toward the ground with no air resistance. The isolated system for which momentum is conserved is (a) the ball (b) the Earth (c) the ball and the Earth (d) impossible to determine.

Quick Quiz 9.4 A car and a large truck traveling at the same speed make a head-on collision and stick together. Which vehicle experiences the larger change in the magnitude of momentum? (a) the car (b) the truck (c) The change in the magnitude of momentum is the same for both. (d) impossible to determine.

Example 9.1 The Archer

Interactive

Let us consider the situation proposed at the beginning of this section. A 60-kg archer stands at rest on frictionless ice and fires a 0.50-kg arrow horizontally at 50 m/s (Fig. 9.2). With what velocity does the archer move across the ice after firing the arrow?

Solution We *cannot* solve this problem using Newton's second law, $\Sigma \mathbf{F} = m\mathbf{a}$, because we have no information about the force on the arrow or its acceleration. We *cannot* solve this problem using an energy approach because we do not know how much work is done in pulling the bow back or how much potential energy is stored in the bow. However, we *can* solve this problem very easily with conservation of momentum.

Let us take the system to consist of the archer (including the bow) and the arrow. The system is not isolated because the gravitational force and the normal force act on the system. However, these forces are vertical and perpendicular to the motion of the system. Therefore, there are no external forces in the horizontal direction, and we can consider the system to be isolated in terms of momentum components in this direction.

The total horizontal momentum of the system before the arrow is fired is zero ($m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = 0$), where the archer is particle 1 and the arrow is particle 2. Therefore, the total horizontal momentum after the arrow is fired must be zero; that is,

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = 0$$

We choose the direction of firing of the arrow as the positive x direction. With $m_1 = 60$ kg, $m_2 = 0.50$ kg, and $\mathbf{v}_{2f} = 50\hat{\mathbf{i}}$ m/s, solving for \mathbf{v}_{1f} , we find the recoil velocity of the archer to be

$$\mathbf{v}_{1f} = -\frac{m_2}{m_1} \mathbf{v}_{2f} = -\left(\frac{0.50 \text{ kg}}{60 \text{ kg}}\right) (50\hat{\mathbf{i}} \text{ m/s}) = -0.42\hat{\mathbf{i}} \text{ m/s}$$

The negative sign for \mathbf{v}_{1f} indicates that the archer is moving to the left after the arrow is fired, in the direction opposite

the direction of motion of the arrow, in accordance with Newton's third law. Because the archer is much more massive than the arrow, his acceleration and consequent velocity are much smaller than the acceleration and velocity of the arrow.

What If? What if the arrow were shot in a direction that makes an angle θ with the horizontal? How will this change the recoil velocity of the archer?

Answer The recoil velocity should decrease in magnitude because only a component of the velocity is in the x direction.



Figure 9.2 (Example 9.1) An archer fires an arrow horizontally to the right. Because he is standing on frictionless ice, he will begin to slide to the left across the ice.

If the arrow were shot straight up, for example, there would be no recoil at all—the archer would just be pressed down into the ice because of the firing of the arrow.

Only the x component of the momentum of the arrow should be used in a conservation of momentum statement, because momentum is only conserved in the x direction. In the y direction, the normal force from the ice and the gravitational force are external influences on the system. Conservation of momentum in the x direction gives us

$$m_1 v_{1f} + m_2 v_{2f} \cos \theta = 0$$

leading to

$$v_{1f} = -\frac{m_2}{m_1} v_{2f} \cos \theta$$

For $\theta = 0$, $\cos \theta = 1$ and this reduces to the value when the arrow is fired horizontally. For nonzero values of θ , the cosine function is less than 1 and the recoil velocity is less than the value calculated for $\theta = 0$. If $\theta = 90^\circ$, $\cos \theta = 0$, and there is no recoil velocity v_{1f} , as we argued conceptually.

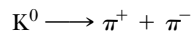


At the Interactive Worked Example link at <http://www.pse6.com>, you can change the mass of the archer and the mass and speed of the arrow.

Example 9.2 Breakup of a Kaon at Rest

One type of nuclear particle, called the *neutral kaon* (K^0), breaks up into a pair of other particles called *pions* (π^+ and π^-) that are oppositely charged but equal in mass, as illustrated in Figure 9.3. Assuming the kaon is initially at rest, prove that the two pions must have momenta that are equal in magnitude and opposite in direction.

Solution The breakup of the kaon can be written



If we let \mathbf{p}^+ be the final momentum of the positive pion and \mathbf{p}^- the final momentum of the negative pion, the final momentum of the system consisting of the two pions can be written

$$\mathbf{p}_f = \mathbf{p}^+ + \mathbf{p}^-$$

Because the kaon is at rest before the breakup, we know that $\mathbf{p}_i = 0$. Because the momentum of the isolated system (the kaon before the breakup, the two pions afterward) is conserved, $\mathbf{p}_i = \mathbf{p}_f = 0$, so that $\mathbf{p}^+ + \mathbf{p}^- = 0$, or

$$\mathbf{p}^+ = -\mathbf{p}^-$$

An important point to learn from this problem is that even though it deals with objects that are very different from those in the preceding example, the physics is identical: *linear momentum is conserved in an isolated system.*

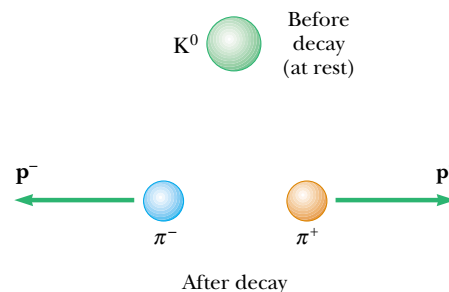


Figure 9.3 (Example 9.2) A kaon at rest breaks up spontaneously into a pair of oppositely charged pions. The pions move apart with momenta that are equal in magnitude but opposite in direction.

9.2 Impulse and Momentum

According to Equation 9.3, the momentum of a particle changes if a net force acts on the particle. Knowing the change in momentum caused by a force is useful in solving some types of problems. To build a better understanding of this important concept, let us assume that a single force \mathbf{F} acts on a particle and that this force may vary with time. According to Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, or

$$d\mathbf{p} = \mathbf{F} dt \quad (9.7)$$

We can integrate² this expression to find the change in the momentum of a particle when the force acts over some time interval. If the momentum of the particle changes from \mathbf{p}_i at time t_i to \mathbf{p}_f at time t_f , integrating Equation 9.7 gives

² Note that here we are integrating force with respect to time. Compare this with our efforts in Chapter 7, where we integrated force with respect to position to find the work done by the force.

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F} dt \quad (9.8)$$

To evaluate the integral, we need to know how the force varies with time. The quantity on the right side of this equation is called the **impulse** of the force \mathbf{F} acting on a particle over the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F} dt \quad (9.9)$$

Equation 9.8 is an important statement known as the **impulse–momentum theorem**:³

The impulse of the force \mathbf{F} acting on a particle equals the change in the momentum of the particle.

This statement is equivalent to Newton's second law. From this definition, we see that impulse is a vector quantity having a magnitude equal to the area under the force–time curve, as described in Figure 9.4a. In this figure, it is assumed that the force varies in time in the general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum—that is, ML/T . Note that impulse is *not* a property of a particle; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\bar{\mathbf{F}} \equiv \frac{1}{\Delta t} \int_{t_i}^{t_f} \mathbf{F} dt \quad (9.10)$$

where $\Delta t = t_f - t_i$. (This is an application of the mean value theorem of calculus.) Therefore, we can express Equation 9.9 as

$$\mathbf{I} \equiv \bar{\mathbf{F}} \Delta t \quad (9.11)$$



Airbags in automobiles have saved countless lives in accidents. The airbag increases the time interval during which the passenger is brought to rest, thereby decreasing the force on (and resultant injury to) the passenger.

Impulse of a force

Impulse–momentum theorem

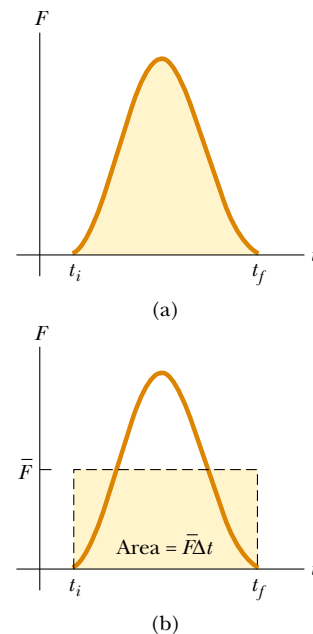


Figure 9.4 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force-versus-time curve. (b) In the time interval Δt , the time-averaged force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

³ Although we assumed that only a single force acts on the particle, the impulse–momentum theorem is valid when several forces act; in this case, we replace \mathbf{F} in Equation 9.8 with $\Sigma \mathbf{F}$.

This time-averaged force, shown in Figure 9.4b, can be interpreted as the constant force that would give to the particle in the time interval Δt the same impulse that the time-varying force gives over this same interval.

In principle, if \mathbf{F} is known as a function of time, the impulse can be calculated from Equation 9.9. The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\bar{\mathbf{F}} = \mathbf{F}$ and Equation 9.11 becomes

$$\mathbf{I} = \mathbf{F} \Delta t \quad (9.12)$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions in which the duration of the collision is very short. When this approximation is made, we refer to the force as an *impulsive force*. For example, when a baseball is struck with a bat, the time of the collision is about 0.01 s and the average force that the bat exerts on the ball in this time is typically several thousand newtons. Because this contact force is much greater than the magnitude of the gravitational force, the impulse approximation justifies our ignoring the gravitational forces exerted on the ball and bat. When we use this approximation, it is important to remember that \mathbf{p}_i and \mathbf{p}_f represent the momenta *immediately* before and after the collision, respectively. Therefore, in any situation in which it is proper to use the impulse approximation, the particle moves very little during the collision.

Quick Quiz 9.5 Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. When a constant force is applied to object 1, it accelerates through a distance d . The force is removed from object 1 and is applied to object 2. At the moment when object 2 has accelerated through the same distance d , which statements are true? (a) $p_1 < p_2$ (b) $p_1 = p_2$ (c) $p_1 > p_2$ (d) $K_1 < K_2$ (e) $K_1 = K_2$ (f) $K_1 > K_2$.

Quick Quiz 9.6 Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. When a force is applied to object 1, it accelerates for a time interval Δt . The force is removed from object 1 and is applied to object 2. After object 2 has accelerated for the same time interval Δt , which statements are true? (a) $p_1 < p_2$ (b) $p_1 = p_2$ (c) $p_1 > p_2$ (d) $K_1 < K_2$ (e) $K_1 = K_2$ (f) $K_1 > K_2$.

Quick Quiz 9.7 Rank an automobile dashboard, seatbelt, and airbag in terms of (a) the impulse and (b) the average force they deliver to a front-seat passenger during a collision, from greatest to least.

Example 9.3 Teeing Off

A golf ball of mass 50 g is struck with a club (Fig. 9.5). The force exerted by the club on the ball varies from zero, at the instant before contact, up to some maximum value and then back to zero when the ball leaves the club. Thus, the force–time curve is qualitatively described by Figure 9.4. Assuming that the ball travels 200 m, estimate the magnitude of the impulse caused by the collision.

Solution Let us use Ⓐ to denote the position of the ball when the club first contacts it, Ⓑ to denote the position of the ball when the club loses contact with the ball, and Ⓒ to denote the position of the ball upon landing. Neglecting

air resistance, we can use Equation 4.14 for the range of a projectile:

$$R = x_C = \frac{v_B^2}{g} \sin 2\theta_B$$

Let us assume that the launch angle θ_B is 45° , the angle that provides the maximum range for any given launch velocity. This assumption gives $\sin 2\theta_B = 1$, and the launch velocity of the ball is

$$v_B = \sqrt{Rg} \approx \sqrt{(200 \text{ m})(9.80 \text{ m/s}^2)} = 44 \text{ m/s}$$

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Figure 9.5 (Example 9.3) A golf ball being struck by a club. Note the deformation of the ball due to the large force from the club.

Considering initial and final values of the ball's velocity for the time interval for the collision, $v_i = v_A = 0$ and $v_f = v_B$. Hence, the magnitude of the impulse imparted to the ball is

$$I = \Delta p = mv_B - mv_A = (50 \times 10^{-3} \text{ kg})(44 \text{ m/s}) - 0 \\ = 2.2 \text{ kg} \cdot \text{m/s}$$

What If? What if you were asked to find the average force on the ball during the collision with the club? Can you determine this value?

Answer With the information given in the problem, we cannot find the average force. Considering Equation 9.11, we would need to know the time interval of the collision in order to calculate the average force. If we *assume* that the time interval is 0.01 s as it was for the baseball in the discussion after Equation 9.12, we can estimate the magnitude of the average force:

$$\bar{F} = \frac{I}{\Delta t} = \frac{2.2 \text{ kg} \cdot \text{m/s}}{0.01 \text{ s}} = 2 \times 10^2 \text{ N}$$

where we have kept only one significant figure due to our rough estimate of the time interval.

Example 9.4 How Good Are the Bumpers?

In a particular crash test, a car of mass 1 500 kg collides with a wall, as shown in Figure 9.6. The initial and final velocities of the car are $\mathbf{v}_i = -15.0\hat{\mathbf{i}} \text{ m/s}$ and $\mathbf{v}_f = 2.60\hat{\mathbf{i}} \text{ m/s}$, respectively. If the collision lasts for 0.150 s, find the impulse caused by the collision and the average force exerted on the car.

Solution Let us assume that the force exerted by the wall on the car is large compared with other forces on the car so that we can apply the impulse approximation. Furthermore, we note that the gravitational force and the normal force

exerted by the road on the car are perpendicular to the motion and therefore do not affect the horizontal momentum.

The initial and final momenta of the car are

$$\mathbf{p}_i = m\mathbf{v}_i = (1500 \text{ kg})(-15.0\hat{\mathbf{i}} \text{ m/s}) \\ = -2.25 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}$$

$$\mathbf{p}_f = m\mathbf{v}_f = (1500 \text{ kg})(2.60\hat{\mathbf{i}} \text{ m/s}) \\ = 0.39 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}$$

Hence, the impulse is equal to



Figure 9.6 (Example 9.4) (a) This car's momentum changes as a result of its collision with the wall. (b) In a crash test, much of the car's initial kinetic energy is transformed into energy associated with the damage to the car.

$$\begin{aligned}\mathbf{I} &= \Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = 0.39 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s} \\ &\quad - (-2.25 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}) \\ \mathbf{I} &= 2.64 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}\end{aligned}$$

The average force exerted by the wall on the car is

$$\bar{\mathbf{F}} = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}}{0.150 \text{ s}} = 1.76 \times 10^5 \hat{\mathbf{i}} \text{ N}$$

In this problem, note that the signs of the velocities indicate the reversal of directions. What would the mathematics be describing if both the initial and final velocities had the same sign?

What If? What if the car did not rebound from the wall? Suppose the final velocity of the car is zero and the time interval of the collision remains at 0.150 s. Would this represent a larger or a smaller force by the wall on the car?

Answer In the original situation in which the car rebounds, the force by the wall on the car does two things in the time interval—it (1) stops the car and (2) causes it to move away from the wall at 2.60 m/s after the collision. If the car does not rebound, the force is only doing the first of these, stopping the car. This will require a *smaller* force.

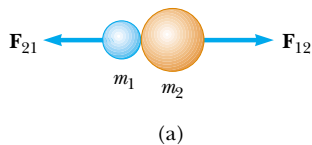
Mathematically, in the case of the car that does not rebound, the impulse is

$$\begin{aligned}\mathbf{I} = \Delta \mathbf{p} &= \mathbf{p}_f - \mathbf{p}_i = 0 - (-2.25 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}) \\ &= 2.25 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}\end{aligned}$$

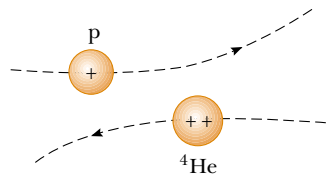
The average force exerted by the wall on the car is

$$\bar{\mathbf{F}} = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{2.25 \times 10^4 \hat{\mathbf{i}} \text{ kg} \cdot \text{m/s}}{0.150 \text{ s}} = 1.50 \times 10^5 \hat{\mathbf{i}} \text{ N}$$

which is indeed smaller than the previously calculated value, as we argued conceptually.



(a)



(b)

Figure 9.7 (a) The collision between two objects as the result of direct contact. (b) The “collision” between two charged particles.

9.3 Collisions in One Dimension

In this section we use the law of conservation of linear momentum to describe what happens when two particles collide. We use the term **collision** to represent an event during which two particles come close to each other and interact by means of forces. The time interval during which the velocities of the particles change from initial to final values is assumed to be short. The interaction forces are assumed to be much greater than any external forces present, so we can use the impulse approximation.

A collision may involve physical contact between two macroscopic objects, as described in Figure 9.7a, but the notion of what we mean by collision must be generalized because “physical contact” on a submicroscopic scale is ill-defined and hence meaningless. To understand this, consider a collision on an atomic scale (Fig. 9.7b), such as the collision of a proton with an alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they repel each other due to the strong electrostatic force between them at close separations and never come into “physical contact.”

When two particles of masses m_1 and m_2 collide as shown in Figure 9.7, the impulsive forces may vary in time in complicated ways, such as that shown in Figure 9.4. Regardless of the complexity of the time behavior of the force of interaction, however, this force is internal to the system of two particles. Thus, the two particles form an isolated system, and the momentum of the system must be conserved. Therefore, the total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.

In contrast, the total kinetic energy of the system of particles may or may not be conserved, depending on the type of collision. In fact, whether or not kinetic energy is conserved is used to classify collisions as either *elastic* or *inelastic*.

An **elastic collision** between two objects is one in which **the total kinetic energy (as well as total momentum) of the system is the same before and after the collision**. Collisions between certain objects in the macroscopic world, such as billiard balls, are only *approximately* elastic because some deformation and loss of kinetic energy take place. For example, you can hear a billiard ball collision, so you know that some of the energy is being transferred away from the system by sound. An elastic collision must be perfectly silent! *Truly* elastic collisions occur between atomic and subatomic particles.

An **inelastic collision** is one in which **the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved)**. Inelastic collisions are of two types. When the colliding objects stick together after the collision, as happens when a meteorite collides with the Earth,

Elastic collision

Inelastic collision

the collision is called **perfectly inelastic**. When the colliding objects do not stick together, but some kinetic energy is lost, as in the case of a rubber ball colliding with a hard surface, the collision is called **inelastic** (with no modifying adverb). When the rubber ball collides with the hard surface, some of the kinetic energy of the ball is lost when the ball is deformed while it is in contact with the surface.

In most collisions, the kinetic energy of the system is *not* conserved because some of the energy is converted to internal energy and some of it is transferred away by means of sound. Elastic and perfectly inelastic collisions are limiting cases; most collisions fall somewhere between them.

In the remainder of this section, we treat collisions in one dimension and consider the two extreme cases—perfectly inelastic and elastic collisions. The important distinction between these two types of collisions is that **momentum of the system is conserved in all collisions, but kinetic energy of the system is conserved only in elastic collisions**.

Perfectly Inelastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities \mathbf{v}_{1i} and \mathbf{v}_{2i} along the same straight line, as shown in Figure 9.8. The two particles collide head-on, stick together, and then move with some common velocity \mathbf{v}_f after the collision. Because the momentum of an isolated system is conserved in *any* collision, we can say that the total momentum before the collision equals the total momentum of the composite system after the collision:

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \quad (9.13)$$

Solving for the final velocity gives

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \quad (9.14)$$

Elastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities \mathbf{v}_{1i} and \mathbf{v}_{2i} along the same straight line, as shown in Figure 9.9. The two particles collide head-on and then leave the collision site with different velocities, \mathbf{v}_{1f} and \mathbf{v}_{2f} . If the collision is elastic, both the momentum and kinetic energy of the system are conserved. Therefore, considering velocities along the horizontal direction in Figure 9.9, we have

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \quad (9.15)$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 \quad (9.16)$$

Because all velocities in Figure 9.9 are either to the left or the right, they can be represented by the corresponding speeds along with algebraic signs indicating direction. We shall indicate v as positive if a particle moves to the right and negative if it moves to the left.

In a typical problem involving elastic collisions, there are two unknown quantities, and Equations 9.15 and 9.16 can be solved simultaneously to find these. An alternative approach, however—one that involves a little mathematical manipulation of Equation 9.16—often simplifies this process. To see how, let us cancel the factor $\frac{1}{2}$ in Equation 9.16 and rewrite it as

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

and then factor both sides:

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \quad (9.17)$$

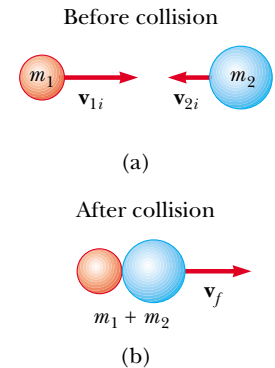
Next, let us separate the terms containing m_1 and m_2 in Equation 9.15 to obtain

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \quad (9.18)$$


PITFALL PREVENTION

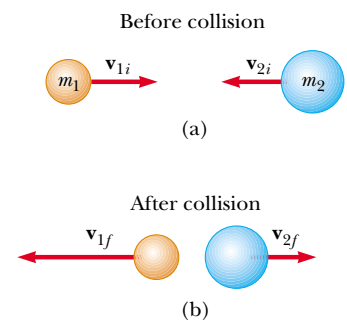
9.2 Inelastic Collisions

Generally, inelastic collisions are hard to analyze unless additional information is provided. This appears in the mathematical representation as having more unknowns than equations.




Active Figure 9.8 Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before collision and (b) after collision.

 **At the Active Figures link at <http://www.pse6.com>, you can adjust the masses and velocities of the colliding objects to see the effect on the final velocity.**



Active Figure 9.9 Schematic representation of an elastic head-on collision between two particles: (a) before collision and (b) after collision.

 **At the Active Figures link at <http://www.pse6.com>, you can adjust the masses and velocities of the colliding objects to see the effect on the final velocities.**

PITFALL PREVENTION

9.3 Not a General Equation

We have spent some effort on deriving Equation 9.19, but remember that it can only be used in a very *specific* situation—a one-dimensional, elastic collision between two objects. The *general* concept is conservation of momentum (and conservation of kinetic energy if the collision is elastic) for an isolated system.

Elastic collision: particle 2 initially at rest

PITFALL PREVENTION

9.4 Momentum and Kinetic Energy in Collisions

Momentum of an isolated system is conserved in *all* collisions. Kinetic energy of an isolated system is conserved *only* in elastic collisions. Why? Because there are several types of energy into which kinetic energy can transform, or be transferred out of the system (so that the system may *not* be isolated in terms of energy during the collision). However, there is only one type of momentum.

To obtain our final result, we divide Equation 9.17 by Equation 9.18 and obtain

$$\begin{aligned} v_{1i} + v_{1f} &= v_{2f} + v_{2i} \\ v_{1i} - v_{2i} &= -(v_{1f} - v_{2f}) \end{aligned} \quad (9.19)$$

This equation, in combination with Equation 9.15, can be used to solve problems dealing with elastic collisions. According to Equation 9.19, the *relative* velocity of the two particles before the collision, $v_{1i} - v_{2i}$, equals the negative of their relative velocity after the collision, $-(v_{1f} - v_{2f})$.

Suppose that the masses and initial velocities of both particles are known. Equations 9.15 and 9.19 can be solved for the final velocities in terms of the initial velocities because there are two equations and two unknowns:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) v_{2i} \quad (9.20)$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i} \quad (9.21)$$

It is important to use the appropriate signs for v_{1i} and v_{2i} in Equations 9.20 and 9.21. For example, if particle 2 is moving to the left initially, then v_{2i} is negative.

Let us consider some special cases. If $m_1 = m_2$, then Equations 9.20 and 9.21 show us that $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, the particles exchange velocities if they have equal masses. This is approximately what one observes in head-on billiard ball collisions—the cue ball stops, and the struck ball moves away from the collision with the same velocity that the cue ball had.

If particle 2 is initially at rest, then $v_{2i} = 0$, and Equations 9.20 and 9.21 become

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} \quad (9.22)$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} \quad (9.23)$$

If m_1 is much greater than m_2 and $v_{2i} = 0$, we see from Equations 9.22 and 9.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. That is, when a very heavy particle collides head-on with a very light one that is initially at rest, the heavy particle continues its motion unaltered after the collision and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle. An example of such a collision would be that of a moving heavy atom, such as uranium, striking a light atom, such as hydrogen.

If m_2 is much greater than m_1 and particle 2 is initially at rest, then $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$. That is, when a very light particle collides head-on with a very heavy particle that is initially at rest, the light particle has its velocity reversed and the heavy one remains approximately at rest.

Quick Quiz 9.8 In a perfectly inelastic one-dimensional collision between two objects, what condition alone is necessary so that *all* of the original kinetic energy of the system is gone after the collision? (a) The objects must have momenta with the same magnitude but opposite directions. (b) The objects must have the same mass. (c) The objects must have the same velocity. (d) The objects must have the same speed, with velocity vectors in opposite directions.

Quick Quiz 9.9 A table-tennis ball is thrown at a stationary bowling ball. The table-tennis ball makes a one-dimensional elastic collision and bounces back along the same line. After the collision, compared to the bowling ball, the table-tennis ball has (a) a larger magnitude of momentum and more kinetic energy (b) a smaller

magnitude of momentum and more kinetic energy (c) a larger magnitude of momentum and less kinetic energy (d) a smaller magnitude of momentum and less kinetic energy (e) the same magnitude of momentum and the same kinetic energy.

Example 9.5 The Executive Stress Reliever

Interactive

An ingenious device that illustrates conservation of momentum and kinetic energy is shown in Figure 9.10. It consists of five identical hard balls supported by strings of equal lengths. When ball 1 is pulled out and released, after the almost-elastic collision between it and ball 2, ball 5 moves out, as shown in Figure 9.10b. If balls 1 and 2 are pulled out and released, balls 4 and 5 swing out, and so forth. Is it ever possible that when ball 1 is released, balls 4 and 5 will swing out on the opposite side and travel with half the speed of ball 1, as in Figure 9.10c?

Solution No, such movement can never occur if we assume the collisions are elastic. The momentum of the system before the collision is mv , where m is the mass of ball 1 and v is its speed just before the collision. After the collision, we would have two balls, each of mass m moving with speed $v/2$. The total momentum of the system after the collision would be $m(v/2) + m(v/2) = mv$. Thus, momentum of the system is conserved. However, the kinetic energy just before the collision is $K_i = \frac{1}{2}mv^2$ and that after the collision is $K_f = \frac{1}{2}m(v/2)^2 + \frac{1}{2}m(v/2)^2 = \frac{1}{4}mv^2$. Thus, kinetic energy of the system is *not* conserved. The only way to have both momentum and kinetic energy conserved is for one ball to move out when one ball is released, two balls to move out when two are released, and so on.

What If? Consider what would happen if balls 4 and 5 are glued together so that they must move together. Now what happens when ball 1 is pulled out and released?

Answer We are now forcing balls 4 and 5 to come out together. We have argued that we cannot conserve both momentum and energy in this case. However, we assumed that ball 1 stopped after striking ball 2. What if we do not make this assumption? Consider the conservation equations with the assumption that ball 1 moves after the collision. For conservation of momentum,

$$p_i = p_f$$

$$mv_{1i} = mv_{1f} + 2mv_{4,5f}$$

where $v_{4,5f}$ refers to the final speed of the ball 4–ball 5 combination. Conservation of kinetic energy gives us

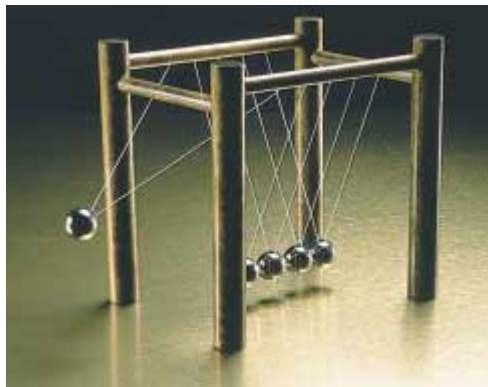
$$K_i = K_f$$

$$\frac{1}{2}mv_{1i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}(2m)v_{4,5f}^2$$

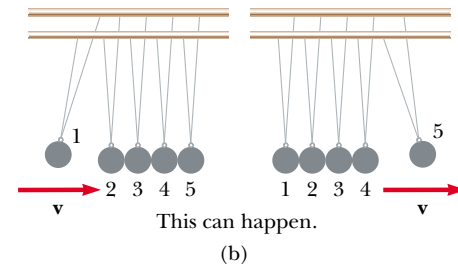
Combining these equations, we find

$$v_{4,5f} = \frac{2}{3}v_{1i} \quad v_{1f} = -\frac{1}{3}v_{1i}$$

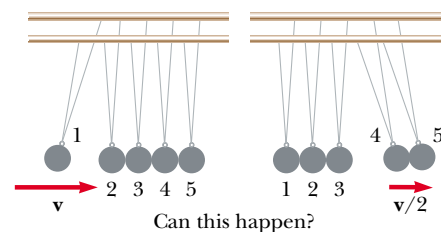
Thus, balls 4 and 5 come out together and ball 1 bounces back from the collision with one third of its original speed.



(a)



(b)



(c)

Figure 9.10 (Example 9.5) An executive stress reliever.



At the Interactive Worked Example link at <http://www.pse6.com>, you can “glue” balls 4 and 5 together to see the situation discussed above.

Example 9.6 Carry Collision Insurance!

An 1 800-kg car stopped at a traffic light is struck from the rear by a 900-kg car, and the two become entangled, moving along the same path as that of the originally moving car. If the smaller car were moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution The phrase “become entangled” tells us that this is a perfectly inelastic collision. We can guess that the final speed is less than 20.0 m/s, the initial speed of the smaller car. The total momentum of the system (the two cars) before the collision must equal the total momentum immediately after the collision because momentum of an isolated system is conserved in any type of collision. The magnitude of the total momentum of the system before the collision is equal to that of the smaller car because the larger car is initially at rest:

$$p_i = m_1 v_i = (900 \text{ kg})(20.0 \text{ m/s}) = 1.80 \times 10^4 \text{ kg} \cdot \text{m/s}$$

After the collision, the magnitude of the momentum of the entangled cars is

$$p_f = (m_1 + m_2)v_f = (2\,700 \text{ kg})v_f$$

Equating the initial and final momenta of the system and solving for v_f , the final velocity of the entangled cars, we have

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{1.80 \times 10^4 \text{ kg} \cdot \text{m/s}}{2\,700 \text{ kg}} = 6.67 \text{ m/s}$$

Because the final velocity is positive, the direction of the final velocity is the same as the velocity of the initially moving car.

What If? Suppose we reverse the masses of the cars—a stationary 900-kg car is struck by a moving 1 800-kg car. Is the final speed the same as before?

Answer Intuitively, we can guess that the final speed will be higher, based on common experiences in driving. Mathematically, this should be the case because the system has a larger momentum if the initially moving car is the more massive one. Solving for the new final velocity, we find

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{(1\,800 \text{ kg})(20.0 \text{ m/s})}{2\,700 \text{ kg}} = 13.3 \text{ m/s}$$

which is indeed higher than the previous final velocity.

Example 9.7 The Ballistic Pendulum

The ballistic pendulum (Fig. 9.11) is an apparatus used to measure the speed of a fast-moving projectile, such as a bullet. A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h . How can we determine the speed of the bullet from a measurement of h ?

Solution Figure 9.11a helps to conceptualize the situation. Let configuration ① be the bullet and block before the collision, and configuration ② be the bullet and block immediately after colliding. The bullet and the block form an isolated system, so we can categorize the collision between them as a conservation of momentum problem. The collision is perfectly inelastic. To analyze the collision, we note that Equation 9.14 gives the speed of the system right after the collision when we assume the impulse approximation. Noting that $v_{2A} = 0$, Equation 9.14 becomes

$$(1) \quad v_B = \frac{m_1 v_{1A}}{m_1 + m_2}$$

For the process during which the bullet–block combination swings upward to height h (ending at configuration ③), we focus on a *different* system—the bullet, the block, and the Earth. This is an isolated system for energy, so we categorize this part of the motion as a conservation of mechanical energy problem:

$$K_B + U_B = K_C + U_C$$

We begin to analyze the problem by finding the total kinetic energy of the system right after the collision:

$$(2) \quad K_B = \frac{1}{2}(m_1 + m_2)v_B^2$$

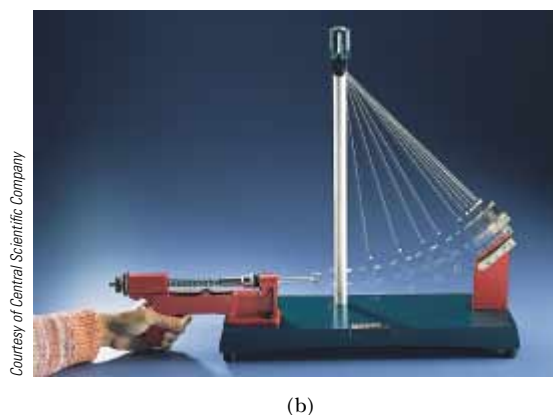
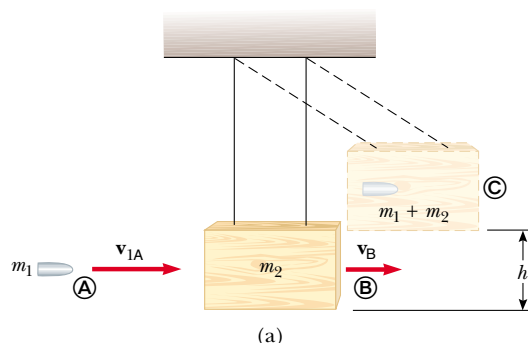


Figure 9.11 (Example 9.7) (a) Diagram of a ballistic pendulum. Note that v_{1A} is the velocity of the bullet just before the collision and v_B is the velocity of the bullet–block system just after the perfectly inelastic collision. (b) Multiframe photograph of a ballistic pendulum used in the laboratory.

Substituting the value of v_B from Equation (1) into Equation (2) gives

$$K_B = \frac{m_1^2 v_{1A}^2}{2(m_1 + m_2)}$$

This kinetic energy immediately after the collision is *less* than the initial kinetic energy of the bullet, as expected in an inelastic collision.

We define the gravitational potential energy of the system for configuration ③ to be zero. Thus, $U_B = 0$ while $U_C = (m_1 + m_2)gh$. Conservation of energy now leads to

$$\frac{m_1^2 v_{1A}^2}{2(m_1 + m_2)} + 0 = 0 + (m_1 + m_2)gh$$

Solving for v_{1A} , we obtain

$$v_{1A} = \left(\frac{m_1 + m_2}{m_1} \right) \sqrt{2gh}$$

To finalize this problem, note that we had to solve this problem in two steps. Each step involved a different system and a different conservation principle. Because the collision was assumed to be perfectly inelastic, some mechanical energy was converted to internal energy. It would have been *incorrect* to equate the initial kinetic energy of the incoming bullet to the final gravitational potential energy of the bullet–block–Earth combination.

Example 9.8 A Two-Body Collision with a Spring

Interactive

A block of mass $m_1 = 1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track collides with a spring attached to a second block of mass $m_2 = 2.10$ kg initially moving to the left with a speed of 2.50 m/s, as shown in Figure 9.12a. The spring constant is 600 N/m.

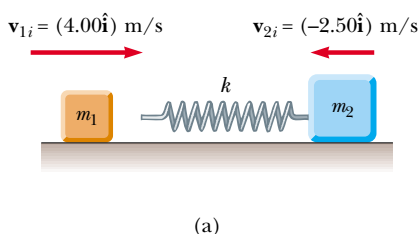
(A) Find the velocities of the two blocks after the collision.

Solution Because the spring force is conservative, no kinetic energy is converted to internal energy during the compression of the spring. Ignoring any sound made when the block hits the spring, we can model the collision as being elastic. Equation 9.15 gives us

$$\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\ (1.60 \text{ kg})(4.00 \text{ m/s}) + (2.10 \text{ kg})(-2.50 \text{ m/s}) &= (1.60 \text{ kg})v_{1f} + (2.10 \text{ kg})v_{2f} \\ (1) \quad 1.15 \text{ kg} \cdot \text{m/s} &= (1.60 \text{ kg})v_{1f} + (2.10 \text{ kg})v_{2f} \end{aligned}$$

Equation 9.19 gives us

$$\begin{aligned} v_{1i} - v_{2i} &= -(v_{1f} - v_{2f}) \\ 4.00 \text{ m/s} - (-2.50 \text{ m/s}) &= -v_{1f} + v_{2f} \\ (2) \quad 6.50 \text{ m/s} &= -v_{1f} + v_{2f} \end{aligned}$$



Multiplying Equation (2) by 1.60 kg gives us

$$(3) \quad 10.4 \text{ kg} \cdot \text{m/s} = -(1.60 \text{ kg})v_{1f} + (1.60 \text{ kg})v_{2f}$$

Adding Equations (1) and (3) allows us to find v_{2f} :

$$11.55 \text{ kg} \cdot \text{m/s} = (3.70 \text{ kg})v_{2f}$$

$$v_{2f} = \frac{11.55 \text{ kg} \cdot \text{m/s}}{3.70 \text{ kg}} = 3.12 \text{ m/s}$$

Now, Equation (2) allows us to find v_{1f} :

$$6.50 \text{ m/s} = -v_{1f} + 3.12 \text{ m/s}$$

$$v_{1f} = -3.38 \text{ m/s}$$

(B) During the collision, at the instant block 1 is moving to the right with a velocity of $+3.00$ m/s, as in Figure 9.12b, determine the velocity of block 2.

Solution Because the momentum of the system of two blocks is conserved *throughout* the collision for the system of two blocks, we have, for *any* instant during the collision,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

We choose the final instant to be that at which block 1 is moving with a velocity of $+3.00$ m/s:

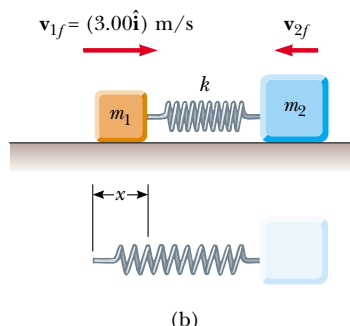


Figure 9.12 (Example 9.8) A moving block approaches a second moving block that is attached to a spring.

$$\begin{aligned}
 (1.60 \text{ kg})(4.00 \text{ m/s}) + (2.10 \text{ kg})(-2.50 \text{ m/s}) \\
 &= (1.60 \text{ kg})(3.00 \text{ m/s}) + (2.10 \text{ kg})v_{2f} \\
 v_{2f} &= -1.74 \text{ m/s}
 \end{aligned}$$

The negative value for v_{2f} means that block 2 is still moving to the left at the instant we are considering.

(C) Determine the distance the spring is compressed at that instant.

Solution To determine the distance that the spring is compressed, shown as x in Figure 9.12b, we can use the principle of conservation of mechanical energy for the system of the spring and two blocks because no friction or other nonconservative forces are acting within the system. We choose the initial configuration of the system to be that existing just before block 1 strikes the spring and the final configuration to be that when block 1 is moving to the right at 3.00 m/s. Thus, we have

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 + 0 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 + \frac{1}{2}kx^2$$

Substituting the given values and the result to part (B) into this expression gives

$$x = 0.173 \text{ m}$$

(D) What is the *maximum* compression of the spring during the collision?

Solution The maximum compression would occur when the two blocks are moving with the same velocity. The conservation of momentum equation for the system can be written

$$m_1v_{1i} + m_2v_{2i} = (m_1 + m_2)v_f$$

where the initial instant is just before the collision and the final instant is when the blocks are moving with the same velocity v_f . Solving for v_f ,

$$\begin{aligned}
 v_f &= \frac{m_1v_{1i} + m_2v_{2i}}{m_1 + m_2} \\
 &= \frac{(1.60 \text{ kg})(4.00 \text{ m/s}) + (2.10 \text{ kg})(-2.50 \text{ m/s})}{1.60 \text{ kg} + 2.10 \text{ kg}} \\
 &= 0.311 \text{ m/s}
 \end{aligned}$$

Now, we apply conservation of mechanical energy between these two instants as in part (C):

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 + 0 = \frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{2}kx^2$$

Substituting the given values into this expression gives

$$x = 0.253 \text{ m}$$



At the *Interactive Worked Example* link at <http://www.pse6.com>, you can change the masses and speeds of the blocks and freeze the motion at the maximum compression of the spring.

Example 9.9 Slowing Down Neutrons by Collisions

In a nuclear reactor, neutrons are produced when an atom splits in a process called *fission*. These neutrons are moving at about 10^7 m/s and must be slowed down to about 10^3 m/s before they take part in another fission event. They are slowed down by passing them through a solid or liquid material called a *moderator*. The slowing-down process involves elastic collisions. Show that a neutron can lose most of its kinetic energy if it collides elastically with a moderator containing light nuclei, such as deuterium (in “heavy water,” D_2O) or carbon (in graphite).

Solution Let us assume that the moderator nucleus of mass m_m is at rest initially and that a neutron of mass m_n and initial speed v_{ni} collides with it head-on. Because these are elastic collisions, both momentum and kinetic energy of the neutron–nucleus system are conserved. Therefore, Equations 9.22 and 9.23 can be applied to the head-on collision of a neutron with a moderator nucleus. We can represent this process by a drawing such as Figure 9.9 with $v_{2i} = 0$.

The initial kinetic energy of the neutron is

$$K_{ni} = \frac{1}{2}m_nv_{ni}^2$$

After the collision, the neutron has kinetic energy $\frac{1}{2}m_nv_{nf}^2$, and we can substitute into this the value for v_{nf} given by

Equation 9.22:

$$K_{nf} = \frac{1}{2}m_nv_{nf}^2 = \frac{1}{2}m_n \left(\frac{m_n - m_m}{m_n + m_m} \right)^2 v_{ni}^2$$

Therefore, the fraction f_n of the initial kinetic energy possessed by the neutron after the collision is

$$(1) \quad f_n = \frac{K_{nf}}{K_{ni}} = \left(\frac{m_n - m_m}{m_n + m_m} \right)^2$$

From this result, we see that the final kinetic energy of the neutron is small when m_m is close to m_n and zero when $m_n = m_m$.

We can use Equation 9.23, which gives the final speed of the particle that was initially at rest, to calculate the kinetic energy of the moderator nucleus after the collision:

$$K_{mf} = \frac{1}{2}m_mv_{mf}^2 = \frac{2m_n^2m_m}{(m_n + m_m)^2} v_{ni}^2$$

Hence, the fraction f_m of the initial kinetic energy transferred to the moderator nucleus is

$$(2) \quad f_m = \frac{K_{mf}}{K_{ni}} = \frac{4m_nm_m}{(m_n + m_m)^2}$$

Because the total kinetic energy of the system is conserved, Equation (2) can also be obtained from Equation (1) with the condition that $f_n + f_m = 1$, so that $f_m = 1 - f_n$.

Suppose that heavy water is used for the moderator. For collisions of the neutrons with deuterium nuclei in D_2O ($m_m = 2m_n$), $f_n = 1/9$ and $f_m = 8/9$. That is, 89% of the

neutron's kinetic energy is transferred to the deuterium nucleus. In practice, the moderator efficiency is reduced because head-on collisions are very unlikely.

How do the results differ when graphite (^{12}C , as found in pencil lead) is used as the moderator?

9.4 Two-Dimensional Collisions

In Section 9.1, we showed that the momentum of a system of two particles is conserved when the system is isolated. For any collision of two particles, this result implies that the momentum in each of the directions x , y , and z is conserved. An important subset of collisions takes place in a plane. The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

where we use three subscripts in these equations to represent, respectively, (1) the identification of the object, (2) initial and final values, and (3) the velocity component.

Let us consider a two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2 , where particle 2 is initially at rest, as in Figure 9.13. After the collision, particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle ϕ with respect to the horizontal. This is called a *glancing* collision. Applying the law of conservation of momentum in component form and noting that the initial y component of the momentum of the two-particle system is zero, we obtain

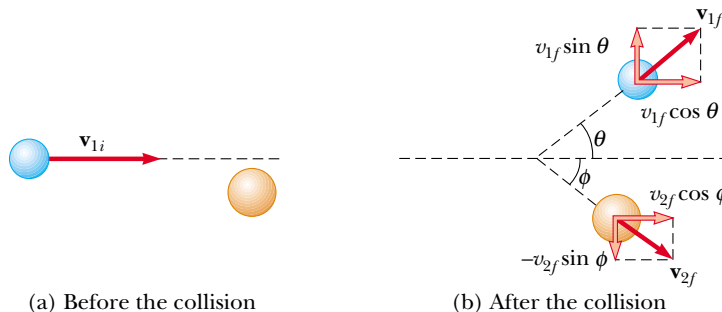
$$m_1 v_{1i} = m_1 v_{1f} \cos \theta + m_2 v_{2f} \cos \phi \quad (9.24)$$

$$0 = m_1 v_{1f} \sin \theta - m_2 v_{2f} \sin \phi \quad (9.25)$$

where the minus sign in Equation 9.25 comes from the fact that after the collision, particle 2 has a y component of velocity that is downward. We now have two independent equations. As long as no more than two of the seven quantities in Equations 9.24 and 9.25 are unknown, we can solve the problem.

If the collision is elastic, we can also use Equation 9.16 (conservation of kinetic energy) with $v_{2i} = 0$ to give

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (9.26)$$




Active Figure 9.13 An elastic glancing collision between two particles.

▲ PITFALL PREVENTION

9.5 Don't Use Equation 9.19

Equation 9.19, relating the initial and final relative velocities of two colliding objects, is only valid for one-dimensional elastic collisions. Do not use this equation when analyzing two-dimensional collisions.

 At the Active Figures link at <http://www.pse6.com>, you can adjust the speed and position of the blue particle and the masses of both particles to see the effects.

Knowing the initial speed of particle 1 and both masses, we are left with four unknowns (v_{1f} , v_{2f} , θ , and ϕ). Because we have only three equations, one of the four remaining quantities must be given if we are to determine the motion after the collision from conservation principles alone.

If the collision is inelastic, kinetic energy is *not* conserved and Equation 9.26 does *not* apply.

PROBLEM-SOLVING HINTS

Two-Dimensional Collisions

The following procedure is recommended when dealing with problems involving two-dimensional collisions between two objects:

- Set up a coordinate system and define your velocities with respect to that system. It is usually convenient to have the x axis coincide with one of the initial velocities.
- In your sketch of the coordinate system, draw and label all velocity vectors and include all the given information.
- Write expressions for the x and y components of the momentum of each object before and after the collision. Remember to include the appropriate signs for the components of the velocity vectors.
- Write expressions for the total momentum of the system in the x direction before and after the collision and equate the two. Repeat this procedure for the total momentum of the system in the y direction.
- If the collision is inelastic, kinetic energy of the system is *not* conserved, and additional information is probably required. If the collision is *perfectly* inelastic, the final velocities of the two objects are equal. Solve the momentum equations for the unknown quantities.
- If the collision is *elastic*, kinetic energy of the system is conserved, and you can equate the total kinetic energy before the collision to the total kinetic energy after the collision to obtain an additional relationship between the velocities.

Example 9.10 Collision at an Intersection

A 1 500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2 500-kg van traveling north at a speed of 20.0 m/s, as shown in Figure 9.14. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).

Solution Let us choose east to be along the positive x direction and north to be along the positive y direction. Before the collision, the only object having momentum in the x direction is the car. Thus, the magnitude of the total initial momentum of the system (car plus van) in the x direction is

$$\sum p_{xi} = (1\,500\text{ kg})(25.0\text{ m/s}) = 3.75 \times 10^4\text{ kg}\cdot\text{m/s}$$

Let us assume that the wreckage moves at an angle θ and speed v_f after the collision. The magnitude of the total momentum in the x direction after the collision is

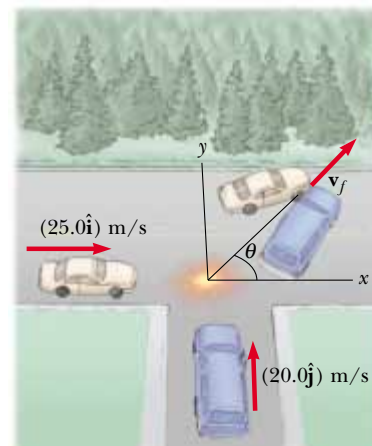


Figure 9.14 (Example 9.10) An eastbound car colliding with a northbound van.

$$\sum p_{xf} = (4\,000\text{ kg})v_f \cos \theta$$

Because the total momentum in the x direction is conserved, we can equate these two equations to obtain

$$(1) \quad 3.75 \times 10^4 \text{ kg} \cdot \text{m/s} = (4\,000\text{ kg})v_f \cos \theta$$

Similarly, the total initial momentum of the system in the y direction is that of the van, and the magnitude of this momentum is $(2\,500\text{ kg})(20.0\text{ m/s}) = 5.00 \times 10^4 \text{ kg} \cdot \text{m/s}$. Applying conservation of momentum to the y direction, we have

$$\sum p_{yi} = \sum p_{yf}$$

$$(2) \quad 5.00 \times 10^4 \text{ kg} \cdot \text{m/s} = (4\,000\text{ kg})v_f \sin \theta$$

If we divide Equation (2) by Equation (1), we obtain

$$\frac{\sin \theta}{\cos \theta} = \tan \theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = 53.1^\circ$$

When this angle is substituted into Equation (2), the value of v_f is

$$v_f = \frac{5.00 \times 10^4 \text{ kg} \cdot \text{m/s}}{(4\,000\text{ kg})\sin 53.1^\circ} = 15.6 \text{ m/s}$$

It might be instructive for you to draw the momentum vectors of each vehicle before the collision and the two vehicles together after the collision.

Example 9.11 Proton-Proton Collision

A proton collides elastically with another proton that is initially at rest. The incoming proton has an initial speed of $3.50 \times 10^5 \text{ m/s}$ and makes a glancing collision with the second proton, as in Figure 9.13. (At close separations, the protons exert a repulsive electrostatic force on each other.) After the collision, one proton moves off at an angle of 37.0° to the original direction of motion, and the second deflects at an angle of ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ .

Solution The pair of protons is an isolated system. Both momentum and kinetic energy of the system are conserved in this glancing elastic collision. Because $m_1 = m_2$, $\theta = 37.0^\circ$, and we are given that $v_{1i} = 3.50 \times 10^5 \text{ m/s}$, Equations 9.24, 9.25, and 9.26 become

$$(1) \quad v_{1f} \cos 37^\circ + v_{2f} \cos \phi = 3.50 \times 10^5 \text{ m/s}$$

$$(2) \quad v_{1f} \sin 37.0^\circ - v_{2f} \sin \phi = 0$$

$$(3) \quad v_{1f}^2 + v_{2f}^2 = (3.50 \times 10^5 \text{ m/s})^2 = 1.23 \times 10^{11} \text{ m}^2/\text{s}^2$$

We rewrite Equations (1) and (2) as follows:

$$v_{2f} \cos \phi = 3.50 \times 10^5 \text{ m/s} - v_{1f} \cos 37.0^\circ$$

$$v_{2f} \sin \phi = v_{1f} \sin 37.0^\circ$$

Now we square these two equations and add them:

$$\begin{aligned} v_{2f}^2 \cos^2 \phi + v_{2f}^2 \sin^2 \phi &= 1.23 \times 10^{11} \text{ m}^2/\text{s}^2 - (7.00 \times 10^5 \text{ m/s})v_{1f} \cos 37.0^\circ \\ &\quad + v_{1f}^2 \cos^2 37.0^\circ + v_{1f}^2 \sin^2 37.0^\circ \\ v_{2f}^2 &= 1.23 \times 10^{11} - (5.59 \times 10^5)v_{1f} + v_{1f}^2 \end{aligned}$$

Substituting into Equation (3) gives

$$\begin{aligned} v_{1f}^2 + [1.23 \times 10^{11} - (5.59 \times 10^5)v_{1f} + v_{1f}^2] &= 1.23 \times 10^{11} \\ 2v_{1f}^2 - (5.59 \times 10^5)v_{1f} &= (2v_{1f} - 5.59 \times 10^5)v_{1f} = 0 \end{aligned}$$

One possibility for the solution of this equation is $v_{1f} = 0$, which corresponds to a head-on collision—the first proton stops and the second continues with the same speed in the same direction. This is not what we want. The other possibility is

$$2v_{1f} - 5.59 \times 10^5 = 0 \quad \longrightarrow \quad v_{1f} = 2.80 \times 10^5 \text{ m/s}$$

From Equation (3),

$$\begin{aligned} v_{2f} &= \sqrt{1.23 \times 10^{11} - v_{1f}^2} = \sqrt{1.23 \times 10^{11} - (2.80 \times 10^5)^2} \\ &= 2.12 \times 10^5 \text{ m/s} \end{aligned}$$

and from Equation (2),

$$\begin{aligned} \phi &= \sin^{-1}\left(\frac{v_{1f} \sin 37.0^\circ}{v_{2f}}\right) = \sin^{-1}\left(\frac{(2.80 \times 10^5) \sin 37.0^\circ}{2.12 \times 10^5}\right) \\ &= 53.0^\circ \end{aligned}$$

It is interesting to note that $\theta + \phi = 90^\circ$. This result is *not* accidental. Whenever two objects of equal mass collide elastically in a glancing collision and one of them is initially at rest, their final velocities are perpendicular to each other. The next example illustrates this point in more detail.

Example 9.12 Billiard Ball Collision

In a game of billiards, a player wishes to sink a target ball in the corner pocket, as shown in Figure 9.15. If the angle to the corner pocket is 35° , at what angle θ is the cue ball deflected? Assume that friction and rotational motion are unimportant and that the collision is elastic. Also assume that all billiard balls have the same mass m .

Solution Let ball 1 be the cue ball and ball 2 be the target ball. Because the target ball is initially at rest, conservation of kinetic energy (Eq. 9.16) for the two-ball system gives

$$\frac{1}{2}m_1v_{1i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

But $m_1 = m_2 = m$, so that

$$(1) \quad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

Applying conservation of momentum to the two-dimensional collision gives

$$(2) \quad m_1 \mathbf{v}_{1i} = m_1 \mathbf{v}_{1f} + m_2 \mathbf{v}_{2f}$$

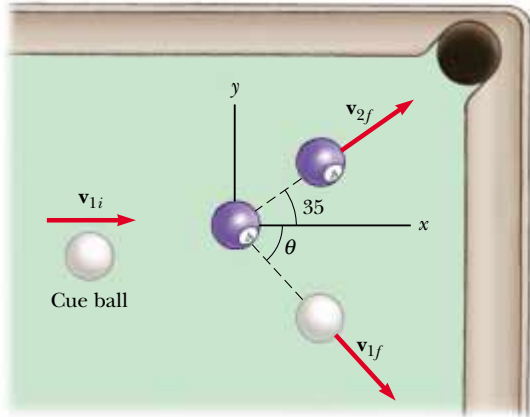


Figure 9.15 (Example 9.12) The cue ball (white) strikes the number 4 ball (blue) and sends it toward the corner pocket.

Note that because $m_1 = m_2 = m$, the masses also cancel in Equation (2). If we square both sides of Equation (2) and use the definition of the dot product of two vectors from Section 7.3, we obtain

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}) = v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

Because the angle between \mathbf{v}_{1f} and \mathbf{v}_{2f} is $\theta + 35^\circ$, $\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} = v_{1f}v_{2f} \cos(\theta + 35^\circ)$, and so

$$(3) \quad v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f} \cos(\theta + 35^\circ)$$

Subtracting Equation (1) from Equation (3) gives

$$0 = 2v_{1f}v_{2f} \cos(\theta + 35^\circ)$$

$$0 = \cos(\theta + 35^\circ)$$

$$\theta + 35^\circ = 90^\circ \quad \text{or} \quad \theta = 55^\circ$$

This result shows that whenever two equal masses undergo a glancing elastic collision and one of them is initially at rest, they move in perpendicular directions after the collision. The same physics describes two very different situations, protons in Example 9.11 and billiard balls in this example.

9.5 The Center of Mass

In this section we describe the overall motion of a mechanical system in terms of a special point called the **center of mass** of the system. The mechanical system can be either a group of particles, such as a collection of atoms in a container, or an extended object, such as a gymnast leaping through the air. We shall see that the center of mass of the system moves as if all the mass of the system were concentrated at that point. Furthermore, if the resultant external force on the system is $\Sigma \mathbf{F}_{\text{ext}}$ and the total mass of the system is M , the center of mass moves with an acceleration given by $\mathbf{a} = \Sigma \mathbf{F}_{\text{ext}}/M$. That is, the system moves as if the resultant external force were applied to a single particle of mass M located at the center of mass. This behavior is independent of other motion, such as rotation or vibration of the system. This is the *particle model* that was introduced in Chapter 2.

Consider a mechanical system consisting of a pair of particles that have different masses and are connected by a light, rigid rod (Fig. 9.16). The position of the center of mass of a system can be described as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass. If a single force is applied at a point on the rod somewhere between the center of mass and the less massive particle, the system rotates clockwise (see Fig. 9.16a). If the force is applied at a point on the rod somewhere between the center of mass and the more massive particle, the system rotates counterclockwise (see Fig. 9.16b). If the force is applied at the center of mass, the system moves in the direction of \mathbf{F} without rotating (see Fig. 9.16c). Thus, the center of mass can be located with this procedure.

The center of mass of the pair of particles described in Figure 9.17 is located on the x axis and lies somewhere between the particles. Its x coordinate is given by

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad (9.27)$$

For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{\text{CM}} = \frac{2}{3}d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.

We can extend this concept to a system of many particles with masses m_i in three dimensions. The x coordinate of the center of mass of n particles is defined to be

$$x_{\text{CM}} \equiv \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M} \quad (9.28)$$

where x_i is the x coordinate of the i th particle. For convenience, we express the total mass as $M \equiv \sum_i m_i$ where the sum runs over all n particles. The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{\text{CM}} \equiv \frac{\sum_i m_i y_i}{M} \quad \text{and} \quad z_{\text{CM}} \equiv \frac{\sum_i m_i z_i}{M} \quad (9.29)$$

The center of mass can also be located by its position vector \mathbf{r}_{CM} . The Cartesian coordinates of this vector are x_{CM} , y_{CM} , and z_{CM} , defined in Equations 9.28 and 9.29. Therefore,

$$\begin{aligned} \mathbf{r}_{\text{CM}} &= x_{\text{CM}}\hat{\mathbf{i}} + y_{\text{CM}}\hat{\mathbf{j}} + z_{\text{CM}}\hat{\mathbf{k}} = \frac{\sum_i m_i x_i \hat{\mathbf{i}} + \sum_i m_i y_i \hat{\mathbf{j}} + \sum_i m_i z_i \hat{\mathbf{k}}}{M} \\ \mathbf{r}_{\text{CM}} &\equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \end{aligned} \quad (9.30)$$

where \mathbf{r}_i is the position vector of the i th particle, defined by

$$\mathbf{r}_i \equiv x_i \hat{\mathbf{i}} + y_i \hat{\mathbf{j}} + z_i \hat{\mathbf{k}}$$

Although locating the center of mass for an extended object is somewhat more cumbersome than locating the center of mass of a system of particles, the basic ideas we have discussed still apply. We can think of an extended object as a system containing a large number of particles (Fig. 9.18). The particle separation is very small, and so the object can be considered to have a continuous mass distribution. By dividing the object into elements of mass Δm_i with coordinates x_i , y_i , z_i , we see that the x coordinate of the center of mass is approximately

$$x_{\text{CM}} \approx \frac{\sum_i x_i \Delta m_i}{M}$$

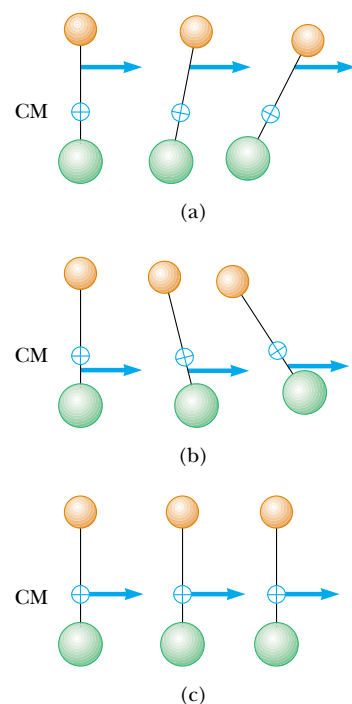
with similar expressions for y_{CM} and z_{CM} . If we let the number of elements n approach infinity, then x_{CM} is given precisely. In this limit, we replace the sum by an integral and Δm_i by the differential element dm :

$$x_{\text{CM}} = \lim_{\Delta m_i \rightarrow 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M} \int x dm \quad (9.31)$$


Likewise, for y_{CM} and z_{CM} we obtain

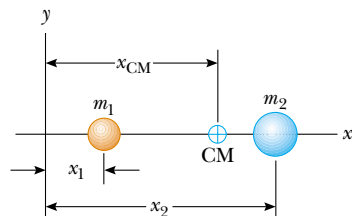
$$y_{\text{CM}} = \frac{1}{M} \int y dm \quad \text{and} \quad z_{\text{CM}} = \frac{1}{M} \int z dm \quad (9.32)$$

We can express the vector position of the center of mass of an extended object in the form




Active Figure 9.16 Two particles of unequal mass are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the less massive particle and the center of mass. (b) The system rotates counterclockwise when a force is applied between the more massive particle and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.

 **At the Active Figures link at <http://www.pse6.com>, you can choose the point at which to apply the force.**



Active Figure 9.17 The center of mass of two particles of unequal mass on the x axis is located at x_{CM} , a point between the particles, closer to the one having the larger mass.

 **At the Active Figures link at <http://www.pse6.com>, you can adjust the masses and positions of the particles to see the effect on the location of the center of mass.**

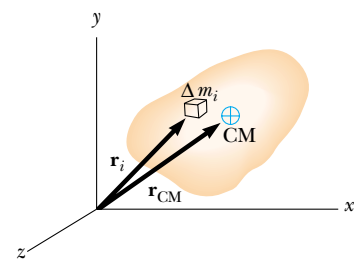


Figure 9.18 An extended object can be considered to be a distribution of small elements of mass Δm_i . The center of mass is located at the vector position \mathbf{r}_{CM} , which has coordinates x_{CM} , y_{CM} , and z_{CM} .

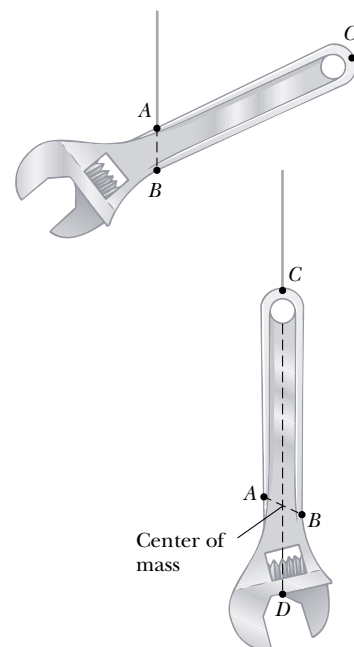


Figure 9.19 An experimental technique for determining the center of mass of a wrench. The wrench is hung freely first from point A and then from point C. The intersection of the two lines AB and CD locates the center of mass.

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \int \mathbf{r} dm \quad (9.33)$$

which is equivalent to the three expressions given by Equations 9.31 and 9.32.

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.⁴ For example, the center of mass of a uniform rod lies in the rod, midway between its ends. The center of mass of a sphere or a cube lies at its geometric center.

The center of mass of an irregularly shaped object such as a wrench can be determined by suspending the object first from one point and then from another. In Figure 9.19, a wrench is hung from point A, and a vertical line AB (which can be established with a plumb bob) is drawn when the wrench has stopped swinging. The wrench is then hung from point C, and a second vertical line CD is drawn. The center of mass is halfway through the thickness of the wrench, under the intersection of these two lines. In general, if the wrench is hung freely from any point, the vertical line through this point must pass through the center of mass.

Because an extended object is a continuous distribution of mass, each small mass element is acted upon by the gravitational force. The net effect of all these forces is equivalent to the effect of a single force $M\mathbf{g}$ acting through a special point, called the **center of gravity**. If \mathbf{g} is constant over the mass distribution, then the center of gravity coincides with the center of mass. If an extended object is pivoted at its center of gravity, it balances in any orientation.

Quick Quiz 9.10 A baseball bat is cut at the location of its center of mass as shown in Figure 9.20. The piece with the smaller mass is (a) the piece on the right (b) the piece on the left (c) Both pieces have the same mass. (d) impossible to determine.



Figure 9.20 (Quick Quiz 9.10) A baseball bat cut at the location of its center of mass.

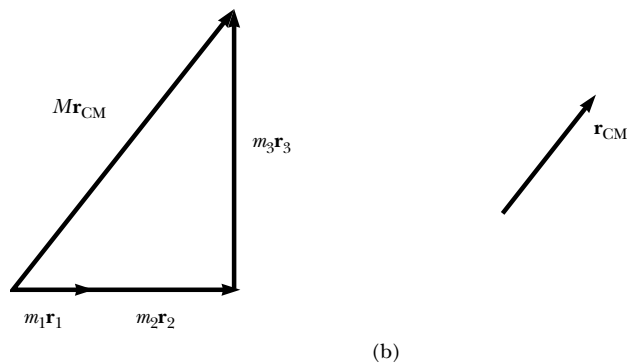
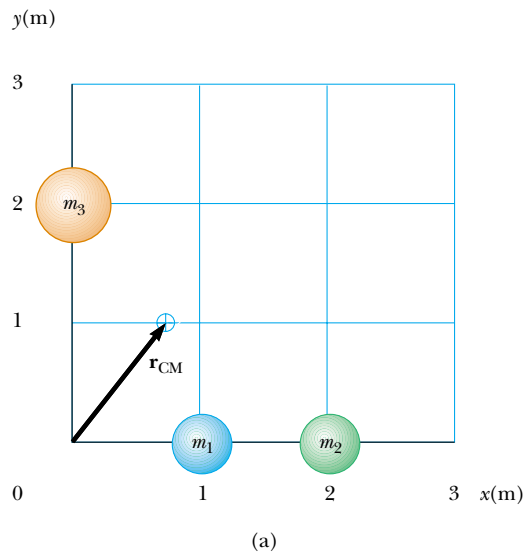
Example 9.13 The Center of Mass of Three Particles

A system consists of three particles located as shown in Figure 9.21a. Find the center of mass of the system.

Solution We set up the problem by labeling the masses of the particles as shown in the figure, with $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg. Using the defining equations for the coordinates of the center of mass and noting that $z_{\text{CM}} = 0$, we obtain

$$\begin{aligned} x_{\text{CM}} &= \frac{\sum m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\ &= \frac{(1.0 \text{ kg})(1.0 \text{ m}) + (1.0 \text{ kg})(2.0 \text{ m}) + (2.0 \text{ kg})(0)}{1.0 \text{ kg} + 1.0 \text{ kg} + 2.0 \text{ kg}} \\ &= \frac{3.0 \text{ kg} \cdot \text{m}}{4.0 \text{ kg}} = 0.75 \text{ m} \end{aligned}$$

⁴ This statement is valid only for objects that have a uniform mass per unit volume.



$$\begin{aligned}
 y_{\text{CM}} &= \frac{\sum_i m_i y_i}{M} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \\
 &= \frac{(1.0 \text{ kg})(0) + (1.0 \text{ kg})(0) + (2.0 \text{ kg})(2.0 \text{ m})}{4.0 \text{ kg}} \\
 &= \frac{4.0 \text{ kg} \cdot \text{m}}{4.0 \text{ kg}} = 1.0 \text{ m}
 \end{aligned}$$

The position vector to the center of mass measured from the origin is therefore

$$\mathbf{r}_{\text{CM}} \equiv x_{\text{CM}} \hat{\mathbf{i}} + y_{\text{CM}} \hat{\mathbf{j}} = (0.75 \hat{\mathbf{i}} + 1.0 \hat{\mathbf{j}}) \text{ m}$$

We can verify this result graphically by adding together $m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3$ and dividing the vector sum by M , the total mass. This is shown in Figure 9.21b.

Figure 9.21 (Example 9.13) (a) Two 1.0-kg particles are located on the x axis and a single 2.0-kg particle is located on the y axis as shown. The vector indicates the location of the system's center of mass. (b) The vector sum of $m_i \mathbf{r}_i$ and the resulting vector for \mathbf{r}_{CM} .

Example 9.14 The Center of Mass of a Rod

(A) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

Solution The rod is shown aligned along the x axis in Figure 9.22, so that $y_{\text{CM}} = z_{\text{CM}} = 0$. Furthermore, if we call the mass per unit length λ (this quantity is called the *linear mass density*), then $\lambda = M/L$ for the uniform rod we assume here. If we divide the rod into elements of length dx , then the mass of each element is $dm = \lambda dx$. Equation 9.31 gives

$$x_{\text{CM}} = \frac{1}{M} \int x dm = \frac{1}{M} \int_0^L x \lambda dx = \frac{\lambda}{M} \left. \frac{x^2}{2} \right|_0^L = \frac{\lambda L^2}{2M}$$

Because $\lambda = M/L$, this reduces to

$$x_{\text{CM}} = \frac{L^2}{2M} \left(\frac{M}{L} \right) = \frac{L}{2}$$

One can also use symmetry arguments to obtain the same result.

(B) Suppose a rod is *nonuniform* such that its mass per unit length varies linearly with x according to the expression

$\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L .

Solution In this case, we replace dm by λdx , where λ is not constant. Therefore, x_{CM} is

$$\begin{aligned}
 x_{\text{CM}} &= \frac{1}{M} \int x dm = \frac{1}{M} \int_0^L x \lambda dx = \frac{1}{M} \int_0^L x \alpha x dx \\
 &= \frac{\alpha}{M} \int_0^L x^2 dx = \frac{\alpha L^3}{3M}
 \end{aligned}$$

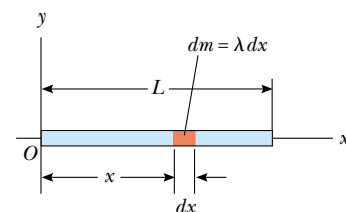


Figure 9.22 (Example 9.14) The geometry used to find the center of mass of a uniform rod.

We can eliminate α by noting that the total mass of the rod is related to α through the relationship

$$M = \int dm = \int_0^L \lambda dx = \int_0^L \alpha x dx = \frac{\alpha L^2}{2}$$

Substituting this into the expression for x_{CM} gives

$$x_{\text{CM}} = \frac{\alpha L^3}{3\alpha L^2/2} = \frac{2}{3}L$$

Example 9.15 The Center of Mass of a Right Triangle

You have been asked to hang a metal sign from a single vertical wire. The sign has the triangular shape shown in Figure 9.23a. The bottom of the sign is to be parallel to the ground. At what distance from the left end of the sign should you attach the support wire?

Solution The wire must be attached at a point directly above the center of gravity of the sign, which is the same as its center of mass because it is in a uniform gravitational field. We assume that the triangular sign has a uniform density and total mass M . Because the sign is a continuous distribution of mass, we must use the integral expression in Equation 9.31 to find the x coordinate of the center of mass.

We divide the triangle into narrow strips of width dx and height y as shown in Figure 9.23b, where y is the height of the hypotenuse of the triangle above the x axis for a given value of x . The mass of each strip is the product of the volume of the strip and the density ρ of the material from which the sign is made: $dm = \rho y t dx$, where t is the thickness of the metal sign. The density of the material is the total mass of the sign divided by its total volume (area of the triangle times thickness), so

$$dm = \rho y t dx = \left(\frac{M}{\frac{1}{2}abt} \right) y t dx = \frac{2My}{ab} dx$$

Using Equation 9.31 to find the x coordinate of the center of mass gives

$$x_{\text{CM}} = \frac{1}{M} \int x dm = \frac{1}{M} \int_0^a x \frac{2My}{ab} dx = \frac{2}{ab} \int_0^a xy dx$$

To proceed further and evaluate the integral, we must express y in terms of x . The line representing the hypotenuse of the triangle in Figure 9.23b has a slope of b/a and passes through the origin, so the equation of this line is $y = (b/a)x$. With this substitution for y in the integral, we have

$$\begin{aligned} x_{\text{CM}} &= \frac{2}{ab} \int_0^a x \left(\frac{b}{a} x \right) dx = \frac{2}{a^2} \int_0^a x^2 dx = \frac{2}{a^2} \left[\frac{x^3}{3} \right]_0^a \\ &= \frac{2}{3}a \end{aligned}$$

Thus, the wire must be attached to the sign at a distance two thirds of the length of the bottom edge from the left end. We could also find the y coordinate of the center of mass of the sign, but this is not needed in order to determine where the wire should be attached.

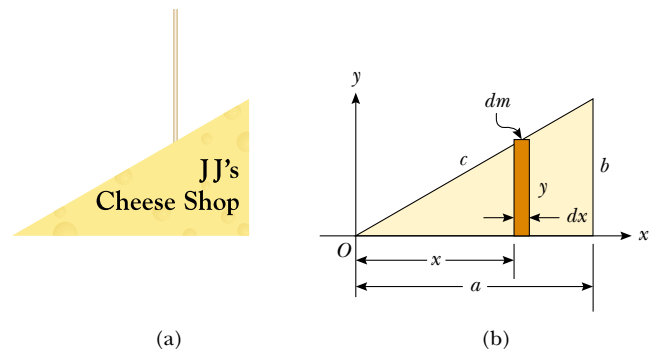


Figure 9.23 (Example 9.15) (a) A triangular sign to be hung from a single wire. (b) Geometric construction for locating the center of mass.

9.6 Motion of a System of Particles

We can begin to understand the physical significance and utility of the center of mass concept by taking the time derivative of the position vector given by Equation 9.30. From Section 4.1 we know that the time derivative of a position vector is by definition a velocity. Assuming M remains constant for a system of particles, that is, no particles enter or leave the system, we obtain the following expression for the **velocity of the center of mass** of the system:

$$\mathbf{v}_{\text{CM}} = \frac{d\mathbf{r}_{\text{CM}}}{dt} = \frac{1}{M} \sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} \quad (9.34)$$

where \mathbf{v}_i is the velocity of the i th particle. Rearranging Equation 9.34 gives

$$M\mathbf{v}_{\text{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\text{tot}} \quad (9.35)$$

Velocity of the center of mass

Total momentum of a system of particles

Therefore, we conclude that the **total linear momentum of the system equals the total mass multiplied by the velocity of the center of mass**. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity \mathbf{v}_{CM} .

If we now differentiate Equation 9.34 with respect to time, we obtain the **acceleration of the center of mass** of the system:

$$\mathbf{a}_{\text{CM}} = \frac{d\mathbf{v}_{\text{CM}}}{dt} = \frac{1}{M} \sum_i m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M} \sum_i m_i \mathbf{a}_i \quad (9.36)$$

Acceleration of the center of mass

Rearranging this expression and using Newton's second law, we obtain

$$M\mathbf{a}_{\text{CM}} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i \quad (9.37)$$

where \mathbf{F}_i is the net force on particle i .

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However, by Newton's third law, the internal force exerted by particle 1 on particle 2, for example, is equal in magnitude and opposite in direction to the internal force exerted by particle 2 on particle 1. Thus, when we sum over all internal forces in Equation 9.37, they cancel in pairs and we find that the net force on the system is caused *only* by external forces. Thus, we can write Equation 9.37 in the form

$$\sum \mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}} \quad (9.38)$$

Newton's second law for a system of particles

That is, **the net external force on a system of particles equals the total mass of the system multiplied by the acceleration of the center of mass**. If we compare this with Newton's second law for a single particle, we see that the particle model that we have used for several chapters can be described in terms of the center of mass:

The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the net external force on the system.

Finally, we see that if the net external force is zero, then from Equation 9.38 it follows that

$$M\mathbf{a}_{\text{CM}} = M \frac{d\mathbf{v}_{\text{CM}}}{dt} = 0$$

so that

$$M\mathbf{v}_{\text{CM}} = \mathbf{p}_{\text{tot}} = \text{constant} \quad (\text{when } \sum \mathbf{F}_{\text{ext}} = 0) \quad (9.39)$$

That is, the total linear momentum of a system of particles is conserved if no net external force is acting on the system. It follows that for an isolated system of particles, both the total momentum and the velocity of the center of mass are constant in time, as shown in Figure 9.24. This is a generalization to a many-particle system of the law of conservation of momentum discussed in Section 9.1 for a two-particle system.

Suppose an isolated system consisting of two or more members is at rest. The center of mass of such a system remains at rest unless acted upon by an external force. For example, consider a system made up of a swimmer standing on a raft, with the system initially at rest. When the swimmer dives horizontally off the raft, the raft moves in the direction opposite to that of the swimmer and the center of mass of the system remains at rest (if we neglect friction between raft and water). Furthermore, the linear momentum of the diver is equal in magnitude to that of the raft, but opposite in direction.

As another example, suppose an unstable atom initially at rest suddenly breaks up into two fragments of masses M_1 and M_2 , with velocities \mathbf{v}_1 and \mathbf{v}_2 , respectively. Because the total momentum of the system before the breakup is zero, the total momentum of the



Figure 9.24 Multiflash photograph showing an overhead view of a wrench moving on a horizontal surface. The white dots are located at the center of mass of the wrench and show that the center of mass moves in a straight line as the wrench rotates.

system after the breakup must also be zero. Therefore, $M_1\mathbf{v}_1 + M_2\mathbf{v}_2 = 0$. If the velocity of one of the fragments is known, the recoil velocity of the other fragment can be calculated.

Quick Quiz 9.11 The vacationers on a cruise ship are eager to arrive at their next destination. They decide to try to speed up the cruise ship by gathering at the bow (the front) and running all at once toward the stern (the back) of the ship. While they are running toward the stern, the speed of the ship is (a) higher than it was before (b) unchanged (c) lower than it was before (d) impossible to determine.

Quick Quiz 9.12 The vacationers in Quick Quiz 9.11 stop running when they reach the stern of the ship. After they have all stopped running, the speed of the ship is (a) higher than it was before they started running (b) unchanged from what it was before they started running (c) lower than it was before they started running (d) impossible to determine.

Conceptual Example 9.16 The Sliding Bear

Suppose you tranquilize a polar bear on a smooth glacier as part of a research effort. How might you estimate the bear's mass using a measuring tape, a rope, and knowledge of your own mass?

Solution Tie one end of the rope around the bear, and then lay out the tape measure on the ice with one end at the bear's original position, as shown in Figure 9.25. Grab hold of the free end of the rope and position yourself as

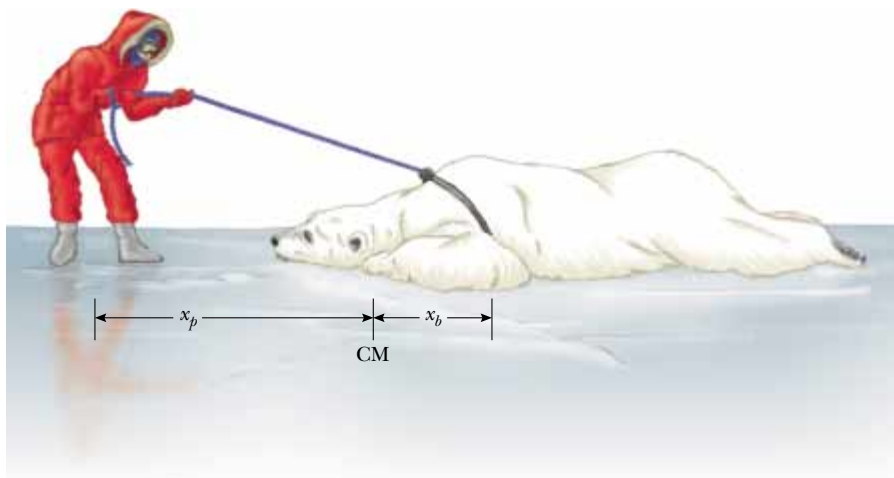


Figure 9.25 (Conceptual Example 9.16) The center of mass of an isolated system remains at rest unless acted on by an external force. How can you determine the mass of the polar bear?

shown, noting your location. Take off your spiked shoes, and pull on the rope hand over hand. Both you and the bear will slide over the ice until you meet. From the tape, observe how far you slide, x_p , and how far the bear slides, x_b . The point where you meet the bear is the fixed

location of the center of mass of the system (bear plus you), and so you can determine the mass of the bear from $m_b x_b = m_p x_p$. (Unfortunately, you cannot return to your spiked shoes and so you are in big trouble if the bear wakes up!)

Conceptual Example 9.17 Exploding Projectile

A projectile fired into the air suddenly explodes into several fragments (Fig. 9.26). What can be said about the motion of the center of mass of the system made up of all the fragments after the explosion?

Solution Neglecting air resistance, the only external force on the projectile is the gravitational force. Thus, if the projectile did not explode, it would continue to move along the parabolic path indicated by the dashed line in Figure 9.26. Because the forces caused by the explosion are internal, they do not affect the motion of the center of mass of the system (the fragments). Thus, after the explosion, the center of mass of the fragments follows the same parabolic path the projectile would have followed if there had been no explosion.

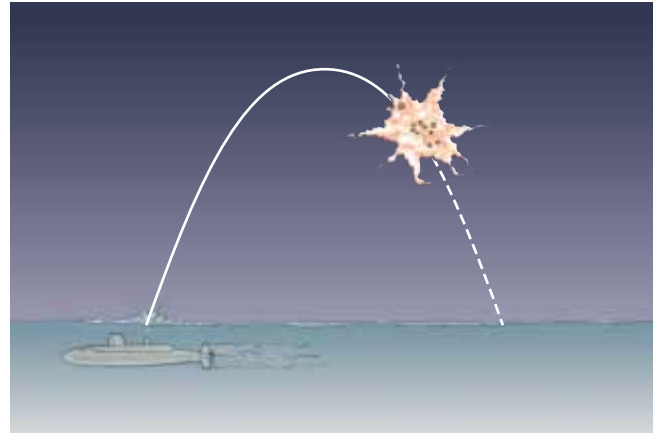


Figure 9.26 (Conceptual Example 9.17) When a projectile explodes into several fragments, the center of mass of the system made up of all the fragments follows the same parabolic path the projectile would have taken had there been no explosion.

Example 9.18 The Exploding Rocket

A rocket is fired vertically upward. At the instant it reaches an altitude of 1 000 m and a speed of 300 m/s, it explodes into three fragments having equal mass. One fragment continues to move upward with a speed of 450 m/s following the explosion. The second fragment has a speed of 240 m/s and is moving east right after the explosion. What is the velocity of the third fragment right after the explosion?

Solution Let us call the total mass of the rocket M ; hence, the mass of each fragment is $M/3$. Because the forces of the explosion are internal to the system and cannot affect its total momentum, the total momentum \mathbf{p}_i of the rocket just before the explosion must equal the total momentum \mathbf{p}_f of the fragments right after the explosion.

Before the explosion,

$$\mathbf{p}_i = M\mathbf{v}_i = M(300\hat{\mathbf{j}} \text{ m/s})$$

After the explosion,

$$\mathbf{p}_f = \frac{M}{3} (240\hat{\mathbf{i}} \text{ m/s}) + \frac{M}{3} (450\hat{\mathbf{j}} \text{ m/s}) + \frac{M}{3} \mathbf{v}_f$$

where \mathbf{v}_f is the unknown velocity of the third fragment. Equating these two expressions (because $\mathbf{p}_i = \mathbf{p}_f$) gives

$$\begin{aligned} \frac{M}{3} \mathbf{v}_f + \frac{M}{3} (240\hat{\mathbf{i}} \text{ m/s}) + \frac{M}{3} (450\hat{\mathbf{j}} \text{ m/s}) \\ = M(300\hat{\mathbf{j}} \text{ m/s}) \end{aligned}$$

$$\mathbf{v}_f = (-240\hat{\mathbf{i}} + 450\hat{\mathbf{j}}) \text{ m/s}$$

What does the sum of the momentum vectors for all the fragments look like?

9.7 Rocket Propulsion

When ordinary vehicles such as cars and locomotives are propelled, the driving force for the motion is friction. In the case of the car, the driving force is the force exerted by the road on the car. A locomotive “pushes” against the tracks; hence, the driving force is the force exerted by the tracks on the locomotive. However, a rocket moving in space has no road or tracks to push against. Therefore, the source of the propulsion of a rocket must be something other than friction. Figure 9.27 is a dramatic photograph of a spacecraft at liftoff. **The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.**



Figure 9.27 At liftoff, enormous thrust is generated by the space shuttle's liquid-fuel engines, aided by the two solid-fuel boosters. This photograph shows the liftoff of the space shuttle *Columbia*, which was lost in a tragic accident during its landing attempt on February 1, 2003 (shortly before this volume went to press).

Rocket propulsion can be understood by first considering a mechanical system consisting of a machine gun mounted on a cart on wheels. As the gun is fired, each bullet receives a momentum $m\mathbf{v}$ in some direction, where \mathbf{v} is measured with respect to a stationary Earth frame. The momentum of the system made up of cart, gun, and bullets must be conserved. Hence, for each bullet fired, the gun and cart must receive a compensating momentum in the opposite direction. That is, the reaction force exerted by the bullet on the gun accelerates the cart and gun, and the cart moves in the direction opposite that of the bullets. If n is the number of bullets fired each second, then the average force exerted on the gun is $\mathbf{F} = nm\mathbf{v}$.

In a similar manner, as a rocket moves in free space, its linear momentum changes when some of its mass is released in the form of ejected gases. **Because the gases are given momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction.** Therefore, the rocket is accelerated as a result of the “push,” or thrust, from the exhaust gases. In free space, the center of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process.⁵

Suppose that at some time t , the magnitude of the momentum of a rocket plus its fuel is $(M + \Delta m)v$, where v is the speed of the rocket relative to the Earth (Fig. 9.28a). Over a short time interval Δt , the rocket ejects fuel of mass Δm , and so at the end of this interval the rocket's speed is $v + \Delta v$, where Δv is the change in speed of the rocket (Fig. 9.28b). If the fuel is ejected with a speed v_e relative to the rocket (the subscript “ e ” stands for *exhaust*, and v_e is usually called the *exhaust speed*), the velocity of the fuel relative to a stationary frame of reference is $v - v_e$. Thus, if we equate the total initial mo-

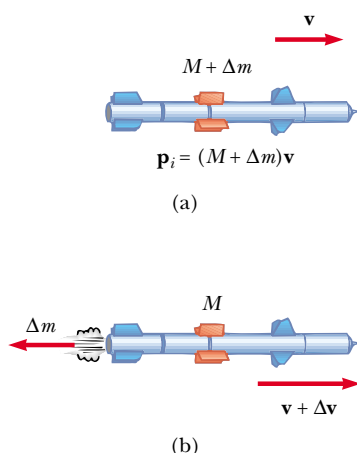


Figure 9.28 Rocket propulsion. (a) The initial mass of the rocket plus all its fuel is $M + \Delta m$ at a time t , and its speed is v . (b) At a time $t + \Delta t$, the rocket's mass has been reduced to M and an amount of fuel Δm has been ejected. The rocket's speed increases by an amount Δv .

⁵ It is interesting to note that the rocket and machine gun represent cases of the reverse of a perfectly inelastic collision: momentum is conserved, but the kinetic energy of the system increases (at the expense of chemical potential energy in the fuel).

mentum of the system to the total final momentum, we obtain

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

where M represents the mass of the rocket and its remaining fuel after an amount of fuel having mass Δm has been ejected. Simplifying this expression gives

$$M\Delta v = v_e\Delta m$$

We also could have arrived at this result by considering the system in the center-of-mass frame of reference, which is a frame having the same velocity as the center of mass of the system. In this frame, the total momentum of the system is zero; therefore, if the rocket gains a momentum $M\Delta v$ by ejecting some fuel, the exhausted fuel obtains a momentum $v_e\Delta m$ in the *opposite* direction, so that $M\Delta v - v_e\Delta m = 0$. If we now take the limit as Δt goes to zero, we let $\Delta v \rightarrow dv$ and $\Delta m \rightarrow dm$. Furthermore, the increase in the exhaust mass dm corresponds to an equal decrease in the rocket mass, so that $dm = -dM$. Note that dM is negative because it represents a decrease in mass, so $-dM$ is a positive number. Using this fact, we obtain

$$M dv = v_e dm = -v_e dM \quad (9.40)$$

We divide the equation by M and integrate, taking the initial mass of the rocket plus fuel to be M_i and the final mass of the rocket plus its remaining fuel to be M_f . This gives

$$\begin{aligned} \int_{v_i}^{v_f} dv &= -v_e \int_{M_i}^{M_f} \frac{dM}{M} \\ v_f - v_i &= v_e \ln \left(\frac{M_i}{M_f} \right) \end{aligned} \quad (9.41)$$

This is the basic expression for rocket propulsion. First, it tells us that the increase in rocket speed is proportional to the exhaust speed v_e of the ejected gases. Therefore, the exhaust speed should be very high. Second, the increase in rocket speed is proportional to the natural logarithm of the ratio M_i/M_f . Therefore, this ratio should be as large as possible, which means that the mass of the rocket without its fuel should be as small as possible and the rocket should carry as much fuel as possible.

The **thrust** on the rocket is the force exerted on it by the ejected exhaust gases. We can obtain an expression for the thrust from Equation 9.40:

$$\text{Thrust} = M \frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right| \quad (9.42)$$

This expression shows us that the thrust increases as the exhaust speed increases and as the rate of change of mass (called the *burn rate*) increases.



Courtesy of NASA

The force from a nitrogen-propelled hand-controlled device allows an astronaut to move about freely in space without restrictive tethers, using the thrust force from the expelled nitrogen.

Expression for rocket propulsion

Example 9.19 A Rocket in Space

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket.

(A) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to half its mass before ignition?

Solution We can guess that the speed we are looking for must be greater than the original speed because the rocket is accelerating. Applying Equation 9.41, we obtain

$$v_f = v_i + v_e \ln \left(\frac{M_i}{M_f} \right)$$

$$\begin{aligned} &= 3.0 \times 10^3 \text{ m/s} + (5.0 \times 10^3 \text{ m/s}) \ln \left(\frac{M_i}{0.5 M_i} \right) \\ &= 6.5 \times 10^3 \text{ m/s} \end{aligned}$$

(B) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Solution Using Equation 9.42,

$$\begin{aligned} \text{Thrust} &= \left| v_e \frac{dM}{dt} \right| = (5.0 \times 10^3 \text{ m/s})(50 \text{ kg/s}) \\ &= 2.5 \times 10^5 \text{ N} \end{aligned}$$

Example 9.20 Fighting a Fire

Two firefighters must apply a total force of 600 N to steady a hose that is discharging water at the rate of 3 600 L/min. Estimate the speed of the water as it exits the nozzle.

Solution The water is exiting at 3 600 L/min, which is 60 L/s. Knowing that 1 L of water has a mass of 1 kg, we estimate that about 60 kg of water leaves the nozzle every second. As the water leaves the hose, it exerts on the hose a thrust that must be counteracted by the 600-N force exerted by the firefighters. So, applying Equation 9.42 gives

$$\begin{aligned}\text{Thrust} &= \left| v_e \frac{dM}{dt} \right| \\ 600 \text{ N} &= |v_e (60 \text{ kg/s})| \\ v_e &= 10 \text{ m/s}\end{aligned}$$

Firefighting is dangerous work. If the nozzle should slip from their hands, the movement of the hose due to the thrust it receives from the rapidly exiting water could injure the firefighters.



Take a practice test for this chapter by clicking on the Practice Test link at <http://www.pse6.com>.

SUMMARY

The **linear momentum** \mathbf{p} of a particle of mass m moving with a velocity \mathbf{v} is

$$\mathbf{p} \equiv m\mathbf{v} \quad (9.2)$$

The law of **conservation of linear momentum** indicates that the total momentum of an isolated system is conserved. If two particles form an isolated system, the momentum of the system is conserved regardless of the nature of the force between them. Therefore, the total momentum of the system at all times equals its initial total momentum, or

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \quad (9.5)$$

The **impulse** imparted to a particle by a force \mathbf{F} is equal to the change in the momentum of the particle:

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F} dt = \Delta\mathbf{p} \quad (9.8, 9.9)$$

This is known as the **impulse-momentum theorem**.

Impulsive forces are often very strong compared with other forces on the system and usually act for a very short time, as in the case of collisions.

When two particles collide, the total momentum of the isolated system before the collision always equals the total momentum after the collision, regardless of the nature of the collision. An **inelastic collision** is one for which the total kinetic energy of the system is not conserved. A **perfectly inelastic collision** is one in which the colliding bodies stick together after the collision. An **elastic collision** is one in which the kinetic energy of the system is conserved.

In a two- or three-dimensional collision, the components of momentum of an isolated system in each of the directions (x , y , and z) are conserved independently.

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \quad (9.30)$$

where $M = \sum_i m_i$ is the total mass of the system and \mathbf{r}_i is the position vector of the i th particle.

The position vector of the center of mass of an extended object can be obtained from the integral expression

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \int \mathbf{r} dm \quad (9.33)$$

The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{\sum_i m_i \mathbf{v}_i}{M} \quad (9.34)$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

Newton's second law applied to a system of particles is

$$\sum \mathbf{F}_{\text{ext}} = M \mathbf{a}_{\text{CM}} \quad (9.38)$$

where \mathbf{a}_{CM} is the acceleration of the center of mass and the sum is over all external forces. The center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system.

QUESTIONS

- Does a large force always produce a larger impulse on an object than a smaller force does? Explain.
- If the speed of a particle is doubled, by what factor is its momentum changed? By what factor is its kinetic energy changed?
- If two particles have equal kinetic energies, are their momenta necessarily equal? Explain.
- While in motion, a pitched baseball carries kinetic energy and momentum. (a) Can we say that it carries a force that it can exert on any object it strikes? (b) Can the baseball deliver more kinetic energy to the object it strikes than the ball carries initially? (c) Can the baseball deliver to the object it strikes more momentum than the ball carries initially? Explain your answers.
- An isolated system is initially at rest. Is it possible for parts of the system to be in motion at some later time? If so, explain how this might occur.
- If two objects collide and one is initially at rest, is it possible for both to be at rest after the collision? Is it possible for one to be at rest after the collision? Explain.
- Explain how linear momentum is conserved when a ball bounces from a floor.
- A bomb, initially at rest, explodes into several pieces. (a) Is linear momentum of the system conserved? (b) Is kinetic energy of the system conserved? Explain.
- A ball of clay is thrown against a brick wall. The clay stops and sticks to the wall. Is the principle of conservation of momentum violated in this example?
- You are standing perfectly still, and then you take a step forward. Before the step your momentum was zero, but afterward you have some momentum. Is the principle of conservation of momentum violated in this case?
- When a ball rolls down an incline, its linear momentum increases. Is the principle of conservation of momentum violated in this process?
- Consider a perfectly inelastic collision between a car and a large truck. Which vehicle experiences a larger change in kinetic energy as a result of the collision?
- A sharpshooter fires a rifle while standing with the butt of the gun against his shoulder. If the forward momentum of a bullet is the same as the backward momentum of the gun, why isn't it as dangerous to be hit by the gun as by the bullet?
- A pole-vaulter falls from a height of 6.0 m onto a foam rubber pad. Can you calculate his speed just before he reaches the pad? Can you calculate the force exerted on him by the pad? Explain.
- Firefighters must apply large forces to hold a fire hose steady (Fig. Q9.15). What factors related to the projection of the water determine the magnitude of the force needed to keep the end of the fire hose stationary?
- A large bed sheet is held vertically by two students. A third student, who happens to be the star pitcher on the baseball team, throws a raw egg at the sheet. Explain why the egg does not break when it hits the sheet, regardless of its initial speed. (If you try this demonstration, make sure the pitcher hits the sheet near its center, and do not allow the egg to fall on the floor after being caught.)






Figure Q9.15

17. A skater is standing still on a frictionless ice rink. Her friend throws a Frisbee straight at her. In which of the following cases is the largest momentum transferred to the skater? (a) The skater catches the Frisbee and holds onto it. (b) The skater catches the Frisbee momentarily, but then drops it vertically downward. (c) The skater catches the Frisbee, holds it momentarily, and throws it back to her friend.
18. In an elastic collision between two particles, does the kinetic energy of each particle change as a result of the collision?
19. Three balls are thrown into the air simultaneously. What is the acceleration of their center of mass while they are in motion?
20. A person balances a meter stick in a horizontal position on the extended index fingers of her right and left hands. She slowly brings the two fingers together. The stick remains balanced and the two fingers always meet at the 50-cm mark regardless of their original positions. (Try it!) Explain.
21. NASA often uses the gravity of a planet to “slingshot” a probe on its way to a more distant planet. The interaction of the planet and the spacecraft is a collision in which the objects do not touch. How can the probe have its speed increased in this manner?
22. The Moon revolves around the Earth. Model its orbit as circular. Is the Moon’s linear momentum conserved? Is its kinetic energy conserved?
23. A raw egg dropped to the floor breaks upon impact. However, a raw egg dropped onto a thick foam rubber cushion from a height of about 1 m rebounds without breaking. Why is this possible? If you try this experiment, be sure to catch the egg after its first bounce.
24. Can the center of mass of an object be located at a position at which there is no mass? If so, give examples.
25. A juggler juggles three balls in a continuous cycle. Any one ball is in contact with his hands for one fifth of the time. Describe the motion of the center of mass of the three balls. What average force does the juggler exert on one ball while he is touching it?
26. Does the center of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
27. Early in the twentieth century, Robert Goddard proposed sending a rocket to the moon. Critics objected that in a vacuum, such as exists between the Earth and the Moon, the gases emitted by the rocket would have nothing to push against to propel the rocket. According to *Scientific American* (January 1975), Goddard placed a gun in a vacuum and fired a blank cartridge from it. (A blank cartridge contains no bullet and fires only the wadding and the hot gases produced by the burning gunpowder.) What happened when the gun was fired?
28. Explain how you could use a balloon to demonstrate the mechanism responsible for rocket propulsion.
29. On the subject of the following positions, state your own view and argue to support it. (a) The best theory of motion is that force causes acceleration. (b) The true measure of a force’s effectiveness is the work it does, and the best theory of motion is that work done on an object changes its energy. (c) The true measure of a force’s effect is impulse, and the best theory of motion is that impulse injected into an object changes its momentum.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*

 = coached solution with hints available at <http://www.pse6.com>  = computer useful in solving problem

 = paired numerical and symbolic problems

Section 9.1 Linear Momentum and its Conservation

1. A 3.00-kg particle has a velocity of $(3.00\hat{i} - 4.00\hat{j})$ m/s. (a) Find its x and y components of momentum. (b) Find the magnitude and direction of its momentum.
2. A 0.100-kg ball is thrown straight up into the air with an initial speed of 15.0 m/s. Find the momentum of the ball (a) at its maximum height and (b) halfway up to its maximum height.
3. How fast can you set the Earth moving? In particular, when you jump straight up as high as you can, what is the order of magnitude of the maximum recoil speed that you give to the Earth? Model the Earth as a perfectly solid object. In your solution, state the physical quantities you take as data and the values you measure or estimate for them.
4. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one

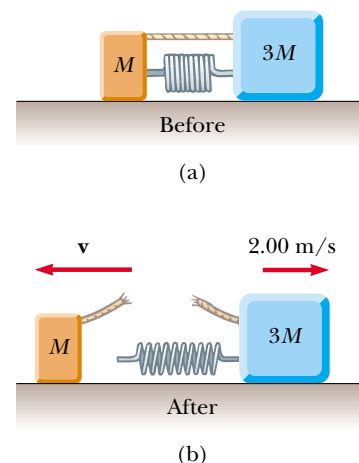


Figure P9.4

of them, and the blocks are pushed together with the spring between them (Fig. P9.4). A cord initially holding the blocks together is burned; after this, the block of mass $3M$ moves to the right with a speed of 2.00 m/s . (a) What is the speed of the block of mass M ? (b) Find the original elastic potential energy in the spring if $M = 0.350\text{ kg}$.

5. (a) A particle of mass m moves with momentum p . Show that the kinetic energy of the particle is $K = p^2/2m$. (b) Express the magnitude of the particle's momentum in terms of its kinetic energy and mass.

Section 9.2 Impulse and Momentum

6. A friend claims that, as long as he has his seatbelt on, he can hold on to a 12.0-kg child in a 60.0 mi/h head-on collision with a brick wall in which the car passenger compartment comes to a stop in 0.050 s . Show that the violent force during the collision will tear the child from his arms. A child should always be in a toddler seat secured with a seat belt in the back seat of a car.
7. An estimated force–time curve for a baseball struck by a bat is shown in Figure P9.7. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

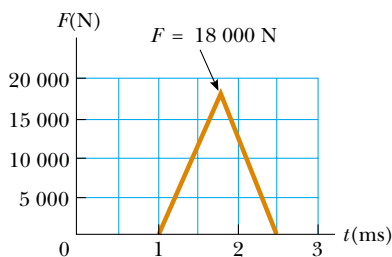


Figure P9.7

8. A ball of mass 0.150 kg is dropped from rest from a height of 1.25 m . It rebounds from the floor to reach a height of 0.960 m . What impulse was given to the ball by the floor?
9. A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle (Fig. P9.9). If the ball is in contact with the wall for 0.200 s , what is the average force exerted by the wall on the ball?

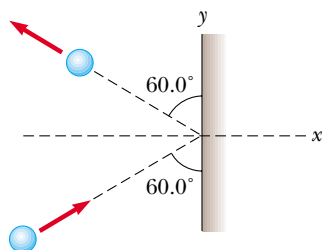


Figure P9.9

10. A tennis player receives a shot with the ball (0.060 kg) traveling horizontally at 50.0 m/s and returns the shot with the ball traveling horizontally at 40.0 m/s in the opposite direction. (a) What is the impulse delivered to the ball by the racquet? (b) What work does the racquet do on the ball?
11. In a slow-pitch softball game, a 0.200-kg softball crosses the plate at 15.0 m/s at an angle of 45.0° below the horizontal. The batter hits the ball toward center field, giving it a velocity of 40.0 m/s at 30.0° above the horizontal. (a) Determine the impulse delivered to the ball. (b) If the force on the ball increases linearly for 4.00 ms , holds constant for 20.0 ms , and then decreases to zero linearly in another 4.00 ms , what is the maximum force on the ball?
12. A professional diver performs a dive from a platform 10 m above the water surface. Estimate the order of magnitude of the average impact force she experiences in her collision with the water. State the quantities you take as data and their values.
13. A garden hose is held as shown in Figure P9.13. The hose is originally full of motionless water. What additional force is necessary to hold the nozzle stationary after the water flow is turned on, if the discharge rate is 0.600 kg/s with a speed of 25.0 m/s ?



Figure P9.13

14. A glider of mass m is free to slide along a horizontal air track. It is pushed against a launcher at one end of the track. Model the launcher as a light spring of force constant k compressed by a distance x . The glider is released from rest. (a) Show that the glider attains a speed of $v = x(k/m)^{1/2}$. (b) Does a glider of large or of small mass attain a greater speed? (c) Show that the impulse imparted to the glider is given by the expression $x(km)^{1/2}$. (d) Is a greater impulse injected into a large or a small mass? (e) Is more work done on a large or a small mass?

Section 9.3 Collisions in One Dimension

15. High-speed stroboscopic photographs show that the head of a golf club of mass 200 g is traveling at 55.0 m/s just before it strikes a 46.0-g golf ball at rest on a tee. After the collision, the club head travels (in the same direction) at 40.0 m/s . Find the speed of the golf ball just after impact.
16. An archer shoots an arrow toward a target that is sliding toward her with a speed of 2.50 m/s on a smooth, slippery

surface. The 22.5-g arrow is shot with a speed of 35.0 m/s and passes through the 300-g target, which is stopped by the impact. What is the speed of the arrow after passing through the target?

17. A 10.0-g bullet is fired into a stationary block of wood ($m = 5.00$ kg). The relative motion of the bullet stops inside the block. The speed of the bullet-plus-wood combination immediately after the collision is 0.600 m/s. What was the original speed of the bullet?

18. A railroad car of mass 2.50×10^4 kg is moving with a speed of 4.00 m/s. It collides and couples with three other coupled railroad cars, each of the same mass as the single car and moving in the same direction with an initial speed of 2.00 m/s. (a) What is the speed of the four cars after the collision? (b) How much mechanical energy is lost in the collision?

19. Four railroad cars, each of mass 2.50×10^4 kg, are coupled together and coasting along horizontal tracks at speed v_i toward the south. A very strong but foolish movie actor, riding on the second car, uncouples the front car and gives it a big push, increasing its speed to 4.00 m/s southward. The remaining three cars continue moving south, now at 2.00 m/s. (a) Find the initial speed of the cars. (b) How much work did the actor do? (c) State the relationship between the process described here and the process in Problem 18.

20. Two blocks are free to slide along the frictionless wooden track ABC shown in Figure P9.20. The block of mass $m_1 = 5.00$ kg is released from A. Protruding from its front end is the north pole of a strong magnet, repelling the north pole of an identical magnet embedded in the back end of the block of mass $m_2 = 10.0$ kg, initially at rest. The two blocks never touch. Calculate the maximum height to which m_1 rises after the elastic collision.

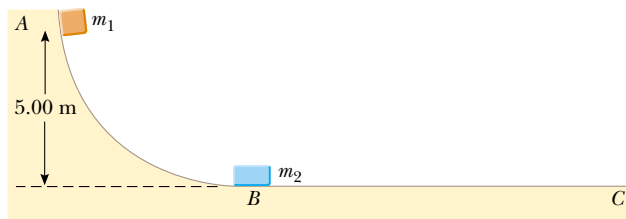


Figure P9.20

21. A 45.0-kg girl is standing on a plank that has a mass of 150 kg. The plank, originally at rest, is free to slide on a frozen lake, which is a flat, frictionless supporting surface. The girl begins to walk along the plank at a constant speed of 1.50 m/s relative to the plank. (a) What is her speed relative to the ice surface? (b) What is the speed of the plank relative to the ice surface?

22. Most of us know intuitively that in a head-on collision between a large dump truck and a subcompact car, you are better off being in the truck than in the car. Why is this? Many people imagine that the collision force exerted on the car is much greater than that experienced by the truck. To substantiate this view, they point out that the car is crushed, whereas the truck is only dented. This idea of unequal

forces, of course, is false. Newton's third law tells us that both objects experience forces of the same magnitude. The truck suffers less damage because it is made of stronger metal. But what about the two drivers? Do they experience the same forces? To answer this question, suppose that each vehicle is initially moving at 8.00 m/s and that they undergo a perfectly inelastic head-on collision. Each driver has mass 80.0 kg. Including the drivers, the total vehicle masses are 800 kg for the car and 4 000 kg for the truck. If the collision time is 0.120 s, what force does the seatbelt exert on each driver?

23. A neutron in a nuclear reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. (a) What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (b) If the initial kinetic energy of the neutron is 1.60×10^{-13} J, find its final kinetic energy and the kinetic energy of the carbon nucleus after the collision. (The mass of the carbon nucleus is nearly 12.0 times the mass of the neutron.)

24. As shown in Figure P9.24, a bullet of mass m and speed v passes completely through a pendulum bob of mass M . The bullet emerges with a speed of $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

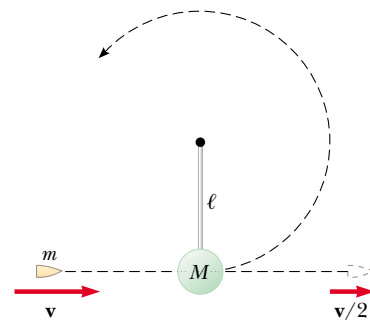


Figure P9.24

25. A 12.0-g wad of sticky clay is hurled horizontally at a 100-g wooden block initially at rest on a horizontal surface. The clay sticks to the block. After impact, the block slides 7.50 m before coming to rest. If the coefficient of friction between the block and the surface is 0.650, what was the speed of the clay immediately before impact?

26. A 7.00-g bullet, when fired from a gun into a 1.00-kg block of wood held in a vise, penetrates the block to a depth of 8.00 cm. **What If?** This block of wood is placed on a frictionless horizontal surface, and a second 7.00-g bullet is fired from the gun into the block. To what depth will the bullet penetrate the block in this case?

27. (a) Three carts of masses 4.00 kg, 10.0 kg, and 3.00 kg move on a frictionless horizontal track with speeds of 5.00 m/s, 3.00 m/s, and 4.00 m/s, as shown in Figure P9.27. Velcro couplers make the carts stick together after colliding. Find the final velocity of the train of three carts. (b) **What If?** Does your answer require that all the carts collide and stick together at the same time? What if they collide in a different order?

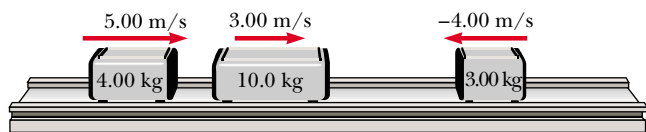


Figure P9.27

Section 9.4 Two-Dimensional Collisions

28. A 90.0-kg fullback running east with a speed of 5.00 m/s is tackled by a 95.0-kg opponent running north with a speed of 3.00 m/s. If the collision is perfectly inelastic, (a) calculate the speed and direction of the players just after the tackle and (b) determine the mechanical energy lost as a result of the collision. Account for the missing energy.

29. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed of 5.00 m/s. After the collision, the orange disk moves along a direction that makes an angle of 37.0° with its initial direction of motion. The velocities of the two disks are perpendicular after the collision. Determine the final speed of each disk.

30. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed v_i . After the collision, the orange disk moves along a direction that makes an angle θ with its initial direction of motion. The velocities of the two disks are perpendicular after the collision. Determine the final speed of each disk.

31. The mass of the blue puck in Figure P9.31 is 20.0% greater than the mass of the green one. Before colliding, the pucks approach each other with momenta of equal magnitudes and opposite directions, and the green puck has an initial speed of 10.0 m/s. Find the speeds of the pucks after the collision if half the kinetic energy is lost during the collision.

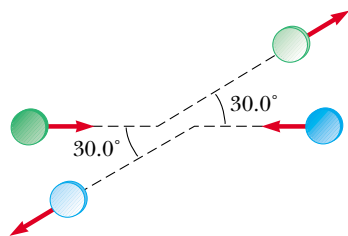


Figure P9.31

32. Two automobiles of equal mass approach an intersection. One vehicle is traveling with velocity 13.0 m/s toward the east, and the other is traveling north with speed v_{2i} . Neither driver sees the other. The vehicles collide in the intersection and stick together, leaving parallel skid marks at an angle of 55.0° north of east. The speed limit for both roads is 35 mi/h, and the driver of the northward-moving vehicle claims he was within the speed limit when the collision occurred. Is he telling the truth?

33. A billiard ball moving at 5.00 m/s strikes a stationary ball of the same mass. After the collision, the first ball moves, at 4.33 m/s, at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision (and ignoring friction and rotational motion), find the struck ball's velocity after the collision.

34. A proton, moving with a velocity of $v_i \hat{i}$, collides elastically with another proton that is initially at rest. If the two protons have equal speeds after the collision, find (a) the speed of each proton after the collision in terms of v_i and (b) the direction of the velocity vectors after the collision.

35. An object of mass 3.00 kg, moving with an initial velocity of $5.00 \hat{i}$ m/s, collides with and sticks to an object of mass 2.00 kg with an initial velocity of $-3.00 \hat{j}$ m/s. Find the final velocity of the composite object.

36. Two particles with masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i . Particle m is traveling to the left, while particle $3m$ is traveling to the right. They undergo an elastic glancing collision such that particle m is moving downward after the collision at right angles from its initial direction. (a) Find the final speeds of the two particles. (b) What is the angle θ at which the particle $3m$ is scattered?

37. An unstable atomic nucleus of mass 17.0×10^{-27} kg initially at rest disintegrates into three particles. One of the particles, of mass 5.00×10^{-27} kg, moves along the y axis with a speed of 6.00×10^6 m/s. Another particle, of mass 8.40×10^{-27} kg, moves along the x axis with a speed of 4.00×10^6 m/s. Find (a) the velocity of the third particle and (b) the total kinetic energy increase in the process.

Section 9.5 The Center of Mass

38. Four objects are situated along the y axis as follows: a 2.00 kg object is at +3.00 m, a 3.00-kg object is at +2.50 m, a 2.50-kg object is at the origin, and a 4.00-kg object is at -0.500 m. Where is the center of mass of these objects?

39. A water molecule consists of an oxygen atom with two hydrogen atoms bound to it (Fig. P9.39). The angle between the two bonds is 106° . If the bonds are 0.100 nm long, where is the center of mass of the molecule?

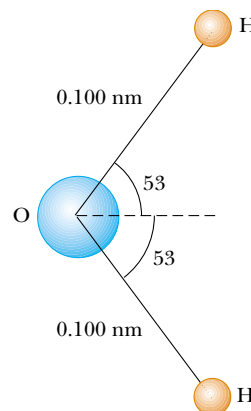


Figure P9.39

40. The mass of the Earth is 5.98×10^{24} kg, and the mass of the Moon is 7.36×10^{22} kg. The distance of separation, measured between their centers, is 3.84×10^8 m. Locate the center of mass of the Earth–Moon system as measured from the center of the Earth.

41. A uniform piece of sheet steel is shaped as in Figure P9.41. Compute the x and y coordinates of the center of mass of the piece.

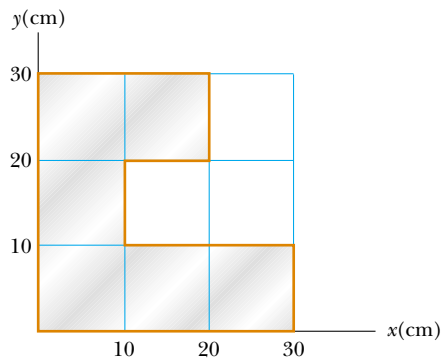


Figure P9.41

42. (a) Consider an extended object whose different portions have different elevations. Assume the free-fall acceleration is uniform over the object. Prove that the gravitational potential energy of the object–Earth system is given by $U_g = Mgy_{\text{CM}}$ where M is the total mass of the object and y_{CM} is the elevation of its center of mass above the chosen reference level. (b) Calculate the gravitational potential energy associated with a ramp constructed on level ground with stone with density $3\,800 \text{ kg/m}^3$ and everywhere 3.60 m wide. In a side view, the ramp appears as a right triangle with height 15.7 m at the top end and base 64.8 m (Figure P9.42).

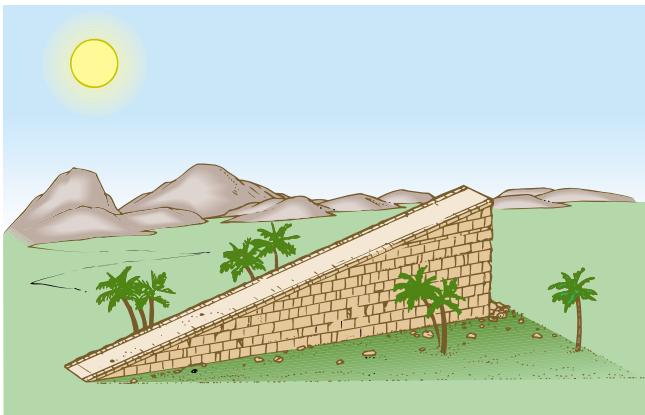


Figure P9.42

43. A rod of length 30.0 cm has linear density (mass-per-length) given by

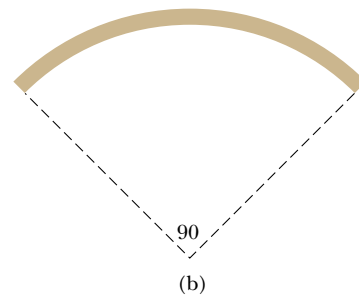
$$\lambda = 50.0 \text{ g/m} + 20.0x \text{ g/m}^2,$$

where x is the distance from one end, measured in meters. (a) What is the mass of the rod? (b) How far from the $x = 0$ end is its center of mass?

44. In the 1968 Olympic Games, University of Oregon jumper Dick Fosbury introduced a new technique of high jumping called the “Fosbury flop.” It contributed to raising the world record by about 30 cm and is presently used by nearly every world-class jumper. In this technique, the jumper goes over the bar face up while arching his back as much as possible, as in Figure P9.44a. This action places his center of mass outside his body, below his back. As his body goes over the bar, his center of mass passes below the bar. Because a given energy input implies a certain elevation for his center of mass, the action of arching his back means his body is higher than if his back were straight. As a model, consider the jumper as a thin uniform rod of length L . When the rod is straight, its center of mass is at its center. Now bend the rod in a circular arc so that it subtends an angle of 90.0° at the center of the arc, as shown in Figure P9.44b. In this configuration, how far outside the rod is the center of mass?



(a)



(b)

Figure P9.44


Section 9.6 Motion of a System of Particles

45. A 2.00-kg particle has a velocity $(2.00\hat{i} - 3.00\hat{j}) \text{ m/s}$, and a 3.00-kg particle has a velocity $(1.00\hat{i} + 6.00\hat{j}) \text{ m/s}$. Find (a) the velocity of the center of mass and (b) the total momentum of the system.
46. Consider a system of two particles in the xy plane: $m_1 = 2.00 \text{ kg}$ is at the location $\mathbf{r}_1 = (1.00\hat{i} + 2.00\hat{j}) \text{ m}$ and has a velocity of $(3.00\hat{i} + 0.500\hat{j}) \text{ m/s}$; $m_2 = 3.00 \text{ kg}$ is at $\mathbf{r}_2 = (-4.00\hat{i} - 3.00\hat{j}) \text{ m}$ and has velocity $(3.00\hat{i} - 2.00\hat{j}) \text{ m/s}$.

(a) Plot these particles on a grid or graph paper. Draw their position vectors and show their velocities. (b) Find the position of the center of mass of the system and mark it on the grid. (c) Determine the velocity of the center of mass and also show it on the diagram. (d) What is the total linear momentum of the system?

47. Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat at rest in still water, 2.70 m away from Juliet, who is in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?
48. A ball of mass 0.200 kg has a velocity of $150\hat{i}$ m/s; a ball of mass 0.300 kg has a velocity of $-0.400\hat{i}$ m/s. They meet in a head-on elastic collision. (a) Find their velocities after the collision. (b) Find the velocity of their center of mass before and after the collision.

Section 9.7 Rocket Propulsion

49.  The first stage of a Saturn V space vehicle consumed fuel and oxidizer at the rate of 1.50×10^4 kg/s, with an exhaust speed of 2.60×10^3 m/s. (a) Calculate the thrust produced by these engines. (b) Find the acceleration of the vehicle just as it lifted off the launch pad on the Earth if the vehicle's initial mass was 3.00×10^6 kg. *Note:* You must include the gravitational force to solve part (b).
50. Model rocket engines are sized by thrust, thrust duration, and total impulse, among other characteristics. A size C5 model rocket engine has an average thrust of 5.26 N, a fuel mass of 12.7 g, and an initial mass of 25.5 g. The duration of its burn is 1.90 s. (a) What is the average exhaust speed of the engine? (b) If this engine is placed in a rocket body of mass 53.5 g, what is the final velocity of the rocket if it is fired in outer space? Assume the fuel burns at a constant rate.
51. A rocket for use in deep space is to be capable of boosting a total load (payload plus rocket frame and engine) of 3.00 metric tons to a speed of 10 000 m/s. (a) It has an engine and fuel designed to produce an exhaust speed of 2 000 m/s. How much fuel plus oxidizer is required? (b) If a different fuel and engine design could give an exhaust speed of 5 000 m/s, what amount of fuel and oxidizer would be required for the same task?
52. *Rocket Science.* A rocket has total mass $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest, turns on its engine at time $t = 0$, and puts out exhaust with relative speed $v_e = 1\,500$ m/s at the constant rate $k = 2.50$ kg/s. The fuel will last for an actual burn time of $330 \text{ kg} / (2.5 \text{ kg/s}) = 132$ s, but define a "projected depletion time" as $T_p = M_i / k = 144$ s. (This would be the burn time if the rocket could use its payload and fuel tanks as fuel, and even the walls of the combustion chamber.) (a) Show that during the burn the velocity of the rocket is given as a function of time by

$$v(t) = -v_e \ln[1 - (t/T_p)]$$

(b) Make a graph of the velocity of the rocket as a function of time for times running from 0 to 132 s. (c) Show that

the acceleration of the rocket is

$$a(t) = v_e / (T_p - t)$$

(d) Graph the acceleration as a function of time. (e) Show that the position of the rocket is

$$x(t) = v_e(T_p - t) \ln[1 - (t/T_p)] + v_e t$$

(f) Graph the position during the burn.

53. An orbiting spacecraft is described not as a "zero-g," but rather as a "microgravity" environment for its occupants and for on-board experiments. Astronauts experience slight lurches due to the motions of equipment and other astronauts, and due to venting of materials from the craft. Assume that a 3 500-kg spacecraft undergoes an acceleration of $2.50 \mu g = 2.45 \times 10^{-5}$ m/s² due to a leak from one of its hydraulic control systems. The fluid is known to escape with a speed of 70.0 m/s into the vacuum of space. How much fluid will be lost in 1 h if the leak is not stopped?

Additional Problems

54. Two gliders are set in motion on an air track. A spring of force constant k is attached to the rear side of one glider. The first glider, of mass m_1 , has velocity \mathbf{v}_1 , and the second glider, of mass m_2 , moves more slowly, with velocity \mathbf{v}_2 , as in Figure P9.54. When m_1 collides with the spring attached to m_2 and compresses the spring to its maximum compression x_{\max} , the velocity of the gliders is \mathbf{v} . In terms of \mathbf{v}_1 , \mathbf{v}_2 , m_1 , m_2 , and k , find (a) the velocity \mathbf{v} at maximum compression, (b) the maximum compression x_{\max} , and (c) the velocity of each glider after m_1 has lost contact with the spring.

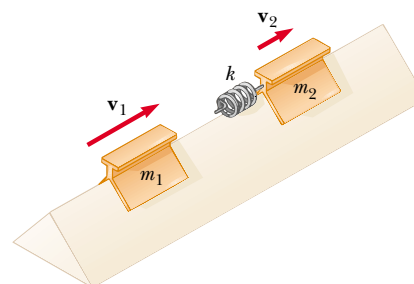


Figure P9.54

55. *Review problem.* A 60.0-kg person running at an initial speed of 4.00 m/s jumps onto a 120-kg cart initially at rest (Figure P9.55). The person slides on the cart's top surface and finally comes to rest relative to the cart. The coefficient of kinetic friction between the person and the cart is 0.400. Friction between the cart and ground can be neglected. (a) Find the final velocity of the person and cart relative to the ground. (b) Find the friction force acting on the person while he is sliding across the top surface of the cart. (c) How long does the friction force act on the person? (d) Find the change in momentum of the person and the change in momentum of the cart. (e) Determine the displacement of the person relative to the ground while he is sliding on the cart. (f) Determine the displacement of the cart relative to the ground while the person is sliding. (g) Find the change in

kinetic energy of the person. (h) Find the change in kinetic energy of the cart. (i) Explain why the answers to (g) and (h) differ. (What kind of collision is this, and what accounts for the loss of mechanical energy?)

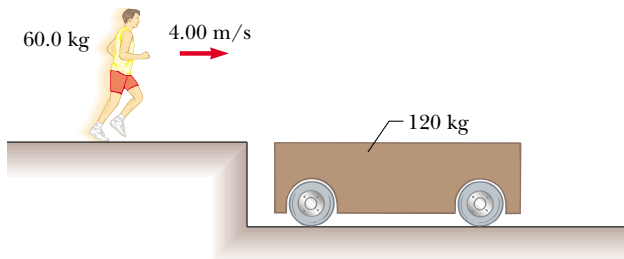


Figure P9.55

56. A golf ball ($m = 46.0$ g) is struck with a force that makes an angle of 45.0° with the horizontal. The ball lands 200 m away on a flat fairway. If the golf club and ball are in contact for 7.00 ms, what is the average force of impact? (Neglect air resistance.)

57. An 80.0-kg astronaut is working on the engines of his ship, which is drifting through space with a constant velocity. The astronaut, wishing to get a better view of the Universe, pushes against the ship and much later finds himself 30.0 m behind the ship. Without a thruster, the only way to return to the ship is to throw his 0.500-kg wrench directly away from the ship. If he throws the wrench with a speed of 20.0 m/s relative to the ship, how long does it take the astronaut to reach the ship?

58. A bullet of mass m is fired into a block of mass M initially at rest at the edge of a frictionless table of height h (Fig. P9.58). The bullet remains in the block, and after impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet.

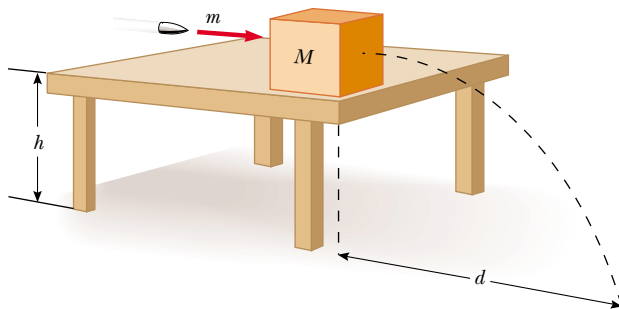


Figure P9.58

59. A 0.500-kg sphere moving with a velocity $(2.00\hat{i} - 3.00\hat{j} + 1.00\hat{k})$ m/s strikes another sphere of mass 1.50 kg moving with a velocity $(-1.00\hat{i} + 2.00\hat{j} - 3.00\hat{k})$ m/s. (a) If the velocity of the 0.500-kg sphere after the collision is $(-1.00\hat{i} + 3.00\hat{j} - 8.00\hat{k})$ m/s, find the final velocity of the 1.50-kg sphere and identify the kind of collision (elastic, inelastic, or perfectly inelastic). (b) If the velocity of the 0.500-kg sphere after the collision is $(-0.250\hat{i} + 0.750\hat{j} -$

$2.00\hat{k})$ m/s, find the final velocity of the 1.50-kg sphere and identify the kind of collision. (c) **What If?** If the velocity of the 0.500-kg sphere after the collision is $(-1.00\hat{i} + 3.00\hat{j} + a\hat{k})$ m/s, find the value of a and the velocity of the 1.50-kg sphere after an elastic collision.

60. A small block of mass $m_1 = 0.500$ kg is released from rest at the top of a curve-shaped frictionless wedge of mass $m_2 = 3.00$ kg, which sits on a frictionless horizontal surface as in Figure P9.60a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right, as in Figure P9.60b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height h of the wedge?

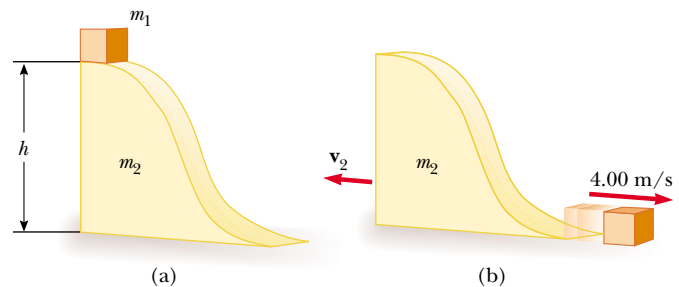


Figure P9.60

61. A bucket of mass m and volume V is attached to a light cart, completely covering its top surface. The cart is given a quick push along a straight, horizontal, smooth road. It is raining, so as the cart cruises along without friction, the bucket gradually fills with water. By the time the bucket is full, its speed is v . (a) What was the initial speed v_i of the cart? Let ρ represent the density of water. (b) **What If?** Assume that when the bucket is half full, it develops a slow leak at the bottom, so that the level of the water remains constant thereafter. Describe qualitatively what happens to the speed of the cart after the leak develops.

62. A 75.0-kg firefighter slides down a pole while a constant friction force of 300 N retards her motion. A horizontal 20.0-kg platform is supported by a spring at the bottom of the pole to cushion the fall. The firefighter starts from rest 4.00 m above the platform, and the spring constant is 4 000 N/m. Find (a) the firefighter's speed just before she collides with the platform and (b) the maximum distance the spring is compressed. (Assume the friction force acts during the entire motion.)

63. George of the Jungle, with mass m , swings on a light vine hanging from a stationary tree branch. A second vine of equal length hangs from the same point, and a gorilla of larger mass M swings in the opposite direction on it. Both vines are horizontal when the primates start from rest at the same moment. George and the gorilla meet at the lowest point of their swings. Each is afraid that one vine will break, so they grab each other and hang on. They swing upward together, reaching a point where the vines make an angle of 35.0° with the vertical. (a) Find the value of the ratio m/M . (b) **What If?** Try this at home. Tie a small magnet and a steel screw to opposite ends of a string. Hold the cen-

ter of the string fixed to represent the tree branch, and reproduce a model of the motions of George and the gorilla. What changes in your analysis will make it apply to this situation? **What If?** Assume the magnet is strong, so that it noticeably attracts the screw over a distance of a few centimeters. Then the screw will be moving faster just before it sticks to the magnet. Does this make a difference?

64. A cannon is rigidly attached to a carriage, which can move along horizontal rails but is connected to a post by a large spring, initially unstretched and with force constant $k = 2.00 \times 10^4 \text{ N/m}$, as in Figure P9.64. The cannon fires a 200-kg projectile at a velocity of 125 m/s directed 45.0° above the horizontal. (a) If the mass of the cannon and its carriage is 5 000 kg, find the recoil speed of the cannon. (b) Determine the maximum extension of the spring. (c) Find the maximum force the spring exerts on the carriage. (d) Consider the system consisting of the cannon, carriage, and shell. Is the momentum of this system conserved during the firing? Why or why not?

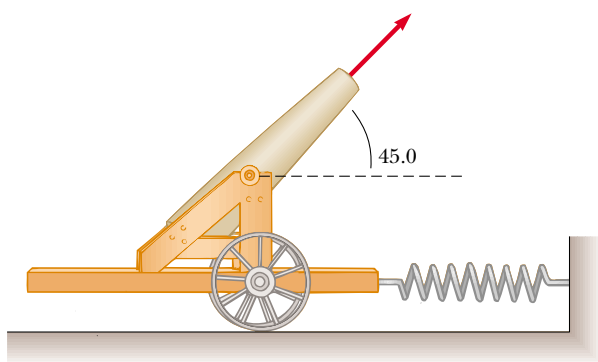


Figure P9.64

65. A student performs a ballistic pendulum experiment using an apparatus similar to that shown in Figure 9.11b. She obtains the following average data: $h = 8.68 \text{ cm}$, $m_1 = 68.8 \text{ g}$, and $m_2 = 263 \text{ g}$. The symbols refer to the quantities in Figure 9.11a. (a) Determine the initial speed v_{1A} of the projectile. (b) The second part of her experiment is to obtain v_{1A} by firing the same projectile horizontally (with the pendulum removed from the path), by measuring its final horizontal position x and distance of fall y (Fig. P9.65). Show that the initial speed of the projectile is related to x and y

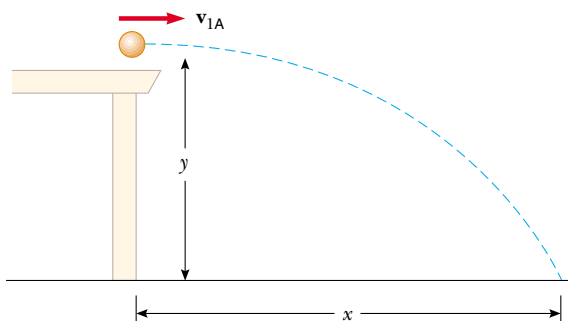


Figure P9.65

through the relation

$$v_{1A} = \frac{x}{\sqrt{2y/g}}$$

What numerical value does she obtain for v_{1A} based on her measured values of $x = 257 \text{ cm}$ and $y = 85.3 \text{ cm}$? What factors might account for the difference in this value compared to that obtained in part (a)?

66. Small ice cubes, each of mass 5.00 g, slide down a frictionless track in a steady stream, as shown in Figure P9.66. Starting from rest, each cube moves down through a net vertical distance of 1.50 m and leaves the bottom end of the track at an angle of 40.0° above the horizontal. At the highest point of its subsequent trajectory, the cube strikes a vertical wall and rebounds with half the speed it had upon impact. If 10.0 cubes strike the wall per second, what average force is exerted on the wall?

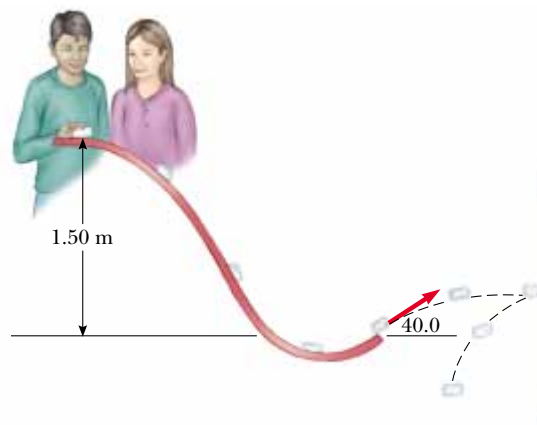


Figure P9.66

67. A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block, as in Figure P9.67. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring with force constant 900 N/m. If the block moves 5.00 cm to the right after impact, find (a) the speed at which the bullet emerges from the block and (b) the mechanical energy converted into internal energy in the collision.

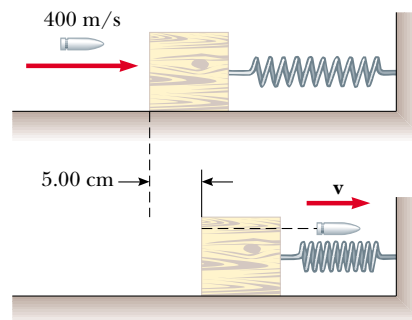



Figure P9.67

68. Consider as a system the Sun with the Earth in a circular orbit around it. Find the magnitude of the change in the velocity of the Sun relative to the center of mass of the

system over a period of 6 months. Neglect the influence of other celestial objects. You may obtain the necessary astronomical data from the endpapers of the book.

- 69. Review problem.** There are (one can say) three coequal theories of motion: Newton's second law, stating that the total force on an object causes its acceleration; the work-kinetic energy theorem, stating that the total work on an object causes its change in kinetic energy; and the impulse-momentum theorem, stating that the total impulse on an object causes its change in momentum. In this problem, you compare predictions of the three theories in one particular case. A 3.00-kg object has velocity $7.00\hat{j}$ m/s. Then, a total force $12.0\hat{i}$ N acts on the object for 5.00 s. (a) Calculate the object's final velocity, using the impulse-momentum theorem. (b) Calculate its acceleration from $\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_i)/\Delta t$. (c) Calculate its acceleration from $\mathbf{a} = \Sigma \mathbf{F}/m$. (d) Find the object's vector displacement from $\Delta \mathbf{r} = \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t^2$. (e) Find the work done on the object from $W = \mathbf{F} \cdot \Delta \mathbf{r}$. (f) Find the final kinetic energy from $\frac{1}{2}mv_f^2 = \frac{1}{2}m\mathbf{v}_f \cdot \mathbf{v}_f$. (g) Find the final kinetic energy from $\frac{1}{2}mv_i^2 + W$.
- 70.**  A rocket has total mass $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest. Its engine is turned on at time $t = 0$, and it puts out exhaust with relative speed $v_e = 1\,500$ m/s at the constant rate 2.50 kg/s. The burn lasts until the fuel runs out, at time $330 \text{ kg}/(2.5 \text{ kg/s}) = 132$ s. Set up and carry out a computer analysis of the motion according to Euler's method. Find (a) the final velocity of the rocket and (b) the distance it travels during the burn.
- 71.** A chain of length L and total mass M is released from rest with its lower end just touching the top of a table, as in Figure P9.71a. Find the force exerted by the table on the chain after the chain has fallen through a distance x , as in Figure P9.71b. (Assume each link comes to rest the instant it reaches the table.)

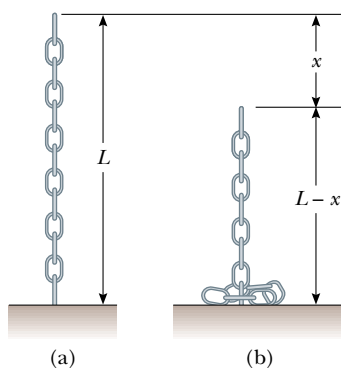


Figure P9.71

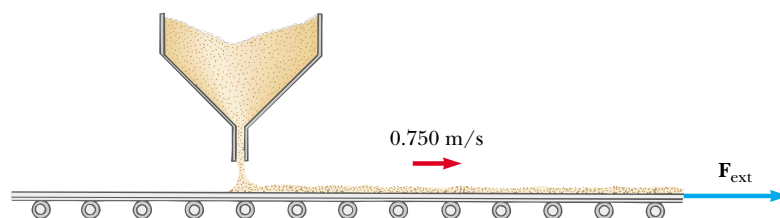


Figure P9.72

- 72.** Sand from a stationary hopper falls onto a moving conveyor belt at the rate of 5.00 kg/s as in Figure P9.72. The conveyor belt is supported by frictionless rollers and moves at a constant speed of 0.750 m/s under the action of a constant horizontal external force \mathbf{F}_{ext} supplied by the motor that drives the belt. Find (a) the sand's rate of change of momentum in the horizontal direction, (b) the force of friction exerted by the belt on the sand, (c) the external force \mathbf{F}_{ext} , (d) the work done by \mathbf{F}_{ext} in 1 s, and (e) the kinetic energy acquired by the falling sand each second due to the change in its horizontal motion. (f) Why are the answers to (d) and (e) different?
- 73.** A golf club consists of a shaft connected to a club head. The golf club can be modeled as a uniform rod of length ℓ and mass m_1 extending radially from the surface of a sphere of radius R and mass m_2 . Find the location of the club's center of mass, measured from the center of the club head.

Answers to Quick Quizzes

- 9.1** (d). Two identical objects ($m_1 = m_2$) traveling at the same speed ($v_1 = v_2$) have the same kinetic energies and the same magnitudes of momentum. It also is possible, however, for particular combinations of masses and velocities to satisfy $K_1 = K_2$ but not $p_1 = p_2$. For example, a 1-kg object moving at 2 m/s has the same kinetic energy as a 4-kg object moving at 1 m/s, but the two clearly do not have the same momenta. Because we have no information about masses and speeds, we cannot choose among (a), (b), or (c).
- 9.2** (b), (c), (a). The slower the ball, the easier it is to catch. If the momentum of the medicine ball is the same as the momentum of the baseball, the speed of the medicine ball must be 1/10 the speed of the baseball because the medicine ball has 10 times the mass. If the kinetic energies are the same, the speed of the medicine ball must be $1/\sqrt{10}$ the speed of the baseball because of the squared speed term in the equation for K . The medicine ball is hardest to catch when it has the same speed as the baseball.
- 9.3** (c). The ball and the Earth exert forces on each other, so neither is an isolated system. We must include both in the system so that the interaction force is internal to the system.
- 9.4** (c). From Equation 9.4, if $\mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$, then it follows that $\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$ and $\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$. While the change in momentum is the same, the change in the velocity is a lot larger for the car!
- 9.5** (c) and (e). Object 2 has a greater acceleration because of its smaller mass. Therefore, it takes less time to travel the distance d . Even though the force applied to objects 1 and 2 is the same, the change in momentum is less for object 2 because Δt is smaller. The work $W = Fd$ done on

both objects is the same because both F and d are the same in the two cases. Therefore, $K_1 = K_2$.

- 9.6 (b) and (d). The same impulse is applied to both objects, so they experience the same change in momentum. Object 2 has a larger acceleration due to its smaller mass. Thus, the distance that object 2 covers in the time interval Δt is larger than that for object 1. As a result, more work is done on object 2 and $K_2 > K_1$.
- 9.7 (a) All three are the same. Because the passenger is brought from the car's initial speed to a full stop, the change in momentum (equal to the impulse) is the same regardless of what stops the passenger. (b) Dashboard, seatbelt, airbag. The dashboard stops the passenger very quickly in a front-end collision, resulting in a very large force. The seatbelt takes somewhat more time, so the force is smaller. Used along with the seatbelt, the airbag can extend the passenger's stopping time further, notably for his head, which would otherwise snap forward.
- 9.8 (a). If all of the initial kinetic energy is transformed, then nothing is moving after the collision. Consequently, the final momentum of the system is necessarily zero and, therefore, the initial momentum of the system must be zero. While (b) and (d) *together* would satisfy the conditions, neither one *alone* does.
- 9.9 (b). Because momentum of the two-ball system is conserved, $\mathbf{p}_{Ti} + 0 = \mathbf{p}_{Tf} + \mathbf{p}_B$. Because the table-tennis ball bounces back from the much more massive bowling ball with approximately the same speed, $\mathbf{p}_{Tf} = -\mathbf{p}_{Ti}$. As a consequence, $\mathbf{p}_B = 2\mathbf{p}_{Ti}$. Kinetic energy can be expressed as $K = p^2/2m$. Because of the much larger mass of the

bowling ball, its kinetic energy is much smaller than that of the table-tennis ball.

- 9.10 (b). The piece with the handle will have less mass than the piece made up of the end of the bat. To see why this is so, take the origin of coordinates as the center of mass before the bat was cut. Replace each cut piece by a small sphere located at the center of mass for each piece. The sphere representing the handle piece is farther from the origin, but the product of less mass and greater distance balances the product of greater mass and less distance for the end piece:



- 9.11 (a). This is the same effect as the swimmer diving off the raft that we just discussed. The vessel-passengers system is isolated. If the passengers all start running one way, the speed of the vessel increases (a *small* amount!) the other way.
- 9.12 (b). Once they stop running, the momentum of the system is the same as it was before they started running—you cannot change the momentum of an isolated system by means of internal forces. In case you are thinking that the passengers could do this over and over to take advantage of the speed increase *while* they are running, remember that they will slow the ship down every time they return to the bow!